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MECHANICS

A TEXTBOOK FOR ENGINEERS

MECHANICS

TEXTBOOK FOR ENGINEERS

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MECHANICS

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PREFACE

The ideals of emphasis on fundamental principles, examples designed to illustrate these principles, and problems to encourage the student to use logical thinking and suitable mathematical methods in their application were the goal of the first edition. These ideals were carried through the second edition and are retained in this one. Even greater emphasis is placed on the attempt to focus the student's attention on the physical meaning of the operations involved in the solutions and to develop his understanding of mechanics as something more than a collection of convenient formulas and operational methods.

Much of the material has been entirely rewritten for the third edition. Many of the problems have been retained, especially where they were particularly well designed to illustrate the principles newly introduced. New problems have been added throughout the text, and these are generally more difficult than the original ones. They are designed to give the student practice in problems of greater complexity and to afford opportunities to develop his skill in attack and analysis.

The arrangement of the second edition, designed to make it convenient to abridge the course by omitting certain articles and some entire chapters, has been followed. The brief chapter on Application of Forces, which serves to emphasize the highly important idea of the free body, has been extended by the addition of a full chapter on Machines and Friction. A chapter has been added on the Method of Work. The chapter on Graphical Methods has been somewhat enlarged by the addition of more advanced problems. The treatment of forces in space has been extended by the inclusion of space frameworks and by the addition of a simple method of computing moments in space by determinants.

The text is designed for a course of about seven or eight semester hours. Where less time is available, Chaps. 7, 10, and 11 may be omitted without destroying the continuity of the book. Chapters 12 and 13, on functions of plane areas and solids, are not strictly a part of statics, but the material is indispensable to an engineering education. These chapters should be omitted only in cases where the material is covered elsewhere in the curriculum.

The treatment of kinetics has been carefully rearranged and largely rewritten to put greater emphasis on orderly, analytical methods. No "special formulas for special cases" are used. With the exception o Chap. 19, dynamics is treated strictly from the standpoint of th Newtonian frame of reference, and the use of fictitious forces an equivalent systems is carefully avoided. Chapter 19 is devoted to brief, simple treatment of moving frames of reference. The example and problems in it are designed to clear up the student's confusio about dynamic forces, inertia forces, centrifugal forces, and the like This chapter could be omitted without destroying continuity but i should be included by all means unless the student is destined to con tinue formal studies in dynamics.

The author is indebted to Professors S. B. Folk and E. C. Clar for constructive criticisms and suggestions and to Professor C. T West for checking some of the derivations and examples

PERCY W. OTT

COLUMBUS, OHIO
May, 1950

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NOTATION

Symbols frequently used in this book are:

a = linear acceleration; apparent moment arm; radius of circle; distance on figure
$\mathbf{a}$ = a vector of length a
A = area; a force (in a few problems)
dA = infinitesimal increment of area
b = breadth; base of a triangle; a distance
$\mathbf{b}$ = a vector of length b
B = a force (in a few problems)
c = a distance; a constant of the catenary
$\mathbf{c}$ = a vector of length c
C = an integration constant; a force (in a few problems)
d = diameter; a distance; pitch of a screw; distance between parallel axes
$\mathbf{d}$ = a vector of length d
e = base of natural logarithms; a distance; coefficient of restitution
E = modulus of elasticity
F = force; total force of friction
g = acceleration of gravity; a constant, 32.174
gd = gudermannian
h = height; height of triangle
hp = horsepower
H = product of inertia; horizontal force
H_c = product of inertia for axes through center of gravity
H' = product of inertia for inclined axes
H_x = horizontal component parallel to X axis.
H_z = horizontal component parallel to Z axis
$\mathbf{H}$ = horizontal vector of length H
I = moment of inertia. Current in amperes
I_c = moment of inertia for axis through center of gravity
I_x = moment of inertia with respect to X axis
I_y = moment of inertia with respect to Y axis
$I_{\text{max.}}$, I_{min} = maximum and minimum moment of inertia
I' = moment of inertia for inclined axis
J = polar moment of inertia
k = radius of gyration; a constant
k_c = radius of gyration for axis through the center of gravity
K = force which deforms a spring one foot; a constant; a constant of integration
l = length; length of a simple pendulum
l', l'' = length of equivalent simple pendulum
m = mass

M = moment
N = force normal to surface
p = a small weight
P = a force; force which causes acceleration
$\mathbf{P}$ = a force of magnitude P
Q = a force; quantity of liquid
r = radius; amplitude of vibration
$\bar{r}$ = radius to center of gravity
$\mathbf{R}$, R = resultant force; reaction of support
s = length; length of catenary from lowest point
ds = infinitesimal increment of length
$\mathbf{S}$, S = equilibrant, a force
t = time; time of vibration; thickness
t_c = time of complete vibration period
dt = infinitesimal increment of time
Δt = a small increment of time
T = torque
$\mathbf{T}$, T = tension; tension in a cord
U = kinetic energy; work
v = linear velocity
dv = infinitesimal increment of velocity
Δv = small increment of velocity
V = volume
$\mathbf{V}$, V = a vertical force
w = weight per unit length
w' = weight per unit of horizontal distance
W = weight; weight or mass in pounds
x, y, z = coordinates; unknown quantities
dx, dy, dz = infinitesimal increments of x, y, and z
X, Y, Z = coordinate axes
$\bar{x}$, $\bar{y}$, $\bar{z}$ = coordinates of center of gravity
α (alpha) = angular acceleration; angle with X axis; any angle; angle of contact of pulley with belt
β (beta) = angle with vertical axis; any angle
γ (gamma) = angle with Z axis; any angle
Δ (delta) = a small increment
θ (theta) = angular displacement; any angle, usually unknown
$d\theta/dt$ = angular velocity = ω
μ (mu) = coefficient of friction
ρ (rho) = density
ϕ (phi) = an angle; angle of friction
ω (omega) = angular velocity
$d\omega/dt$ = angular acceleration = α
Σ (sigma) = summation

CHAPTER 1

FUNDAMENTAL IDEAS

1. Definitions. Mechanics is the science which treats of the effect of forces on the motion or form of bodies. This science is divided into *statics* and *kinetics.* When the forces which act on a body are so balanced as to cause no change in its motion, the problem of finding the relation of these forces falls under the division of *statics.* When the forces which act on a body cause some change in the magnitude or direction of its motion, the problem of finding the relation of the forces to the mass of the body and to the change of its motion falls under the division of *kinetics.* The term *dynamics* is used by some writers as a synonym for kinetics. Others use dynamics to include both statics and kinetics.

The branch of statics which considers the change of form of elastic solids is called *mechanics of materials*, or *theory of elasticity.* When the ultimate strength and allowable loads on the material are combined with the theory of the ideal elastic solid, the subject is called *strength of materials.* The mechanics of fluids is treated under such titles as *hydraulics*, *hydrodynamics*, *thermodynamics*, and *fluid mechanics.*

This volume deals chiefly with the statics and dynamics of *solid bodies,* which are sufficiently *rigid* that the change of form or dimensions does not appreciably affect the calculations. The subjects of strength of materials and fluid mechanics are left for future study.

In the study of the mechanics of rigid bodies, any body selected for consideration is called a *free body.* This term is a little unfortunate, as *constrained body* might have been more appropriate, but it is in almost universal use. A free body may be anything, any part of a thing, or any collection of things. A problem in rigid-body mechanics involves a situation in which the forces acting on one or more free bodies are to be considered in connection with the motions of the body or bodies. For clarity of presentation and convenience of solution, the force system acting on any free body may be classified by the words *concurrent* and *coplanar* and their opposites. Sometimes the further classifications of *parallel* and *symmetrical* are convenient. There are sometimes arbitrary elements in the classification, the system falling into one class or another depending on the assumptions made, as illustrated below; or, on solution, some of the forces may turn out to be zero, leaving the problem in a simpler class than that assumed at the beginning.

The *principles of mechanics* enable us to state unequivocal relations between and among the forces acting on and the motions (if any) of any free body. In any problem, if enough of the forces and elements of motion are known, the remainder of the forces and elements of motion, called the *unknowns*, can be determined. With each classification of force systems, there is associated an invariable number of unknowns which can be determined. A problem having exactly the right number of unknowns can be carried to a definite solution. One having too many unknowns cannot be solved except in the sense that the unknowns can be stated in terms of the knowns and of other unknowns. A problem with too few unknowns for its class results in absurd answers unless it has been carefully or fortuitously devised.

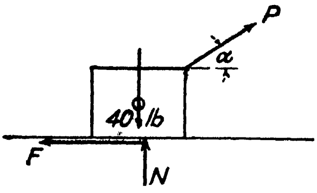


FIG. 1.

2. Illustrations. Figure 1 is an example of a problem in mechanics. The figure represents a mass of 40 pounds on a rough horizontal plane and subjected to a force *P*. The sketch obviously conveys more than an artistic presentation of the idea stated above. The symbol α designates the angle between the direction of the force P and the horizontal. The vertical downward arrow through the center of the body indicates the force exerted on the body by the gravitational attraction between it and the earth. The horizontal arrow to the left under the block represents a force to the left which the plane exerts on the block by an action called *friction*, and the symbol F represents the magnitude of this force. The arrow directed vertically upward under the block represents the *resultant* of all the upward pressures which the plane exerts on the block, and the symbol N represents the magnitude of this force. Dimension lines to show the distance from the left edge of the block, or some other point, to the position of N, and to show the distance from the plane to the point of attachment of the cord exerting the force P might be added if there were any question involving these dimensions.

The addition of these symbols to the simple sketch of the body resting or moving on the plane completes what is called the *free-body* *diagram.* If the position of N is not in question and it is assumed that the body will remain in contact with the plane all along its bottom, the force system depicted in Fig. 1 is classified as a *concurrent, coplanar system.* If it is assumed (or turns out) that the body remains stationary or moves along the plane with constant velocity, the problem of the magnitude of the forces involved and the direction of the force P is a problem of *statics.* If the force P is so large that the friction force F is insufficient to prevent motion, the problem becomes a problem of *dynamics,* and the additional elements of displacement and time must be considered in a complete analysis of the problem. If the position of the force N or the possibility of the block's turning over is open to question, the system of forces becomes a *nonconcurrent, coplanar system,* regardless of whether the problem is one of statics or of dynamics. The assumption that the block will not upset is equivalent to an assumption that the resultant of the forces F and N passes through the point of intersection of the lines of action of P and the vertical gravity force, a point called the *point of concurrency.*

In almost every problem of mechanics some assumptions must be made at the outset. When the problem has been solved, the validity of these assumptions can and should be considered. In Fig. 1, let us assume that the body remains at rest and that the exact location of N is of no interest. The force system classifies as concurrent, coplanar, a system in which the number of unknowns should be exactly two. If any two of the group P, α, F, and N are known, definite values for the other two can be found. If P were 25 pounds and $\alpha = \tan^{-1} 0.75$, F would be 20 pounds and N would be 25 pounds, as will be shown later. But for any two dry surfaces the ratio of the friction force to the normal force has a definite limit, called the *coefficient of friction,* which can be determined only by experiment. If the horizontal plane is of dry, rough-sawn wood and the bottom of the block is of soft rubber, the coefficient of friction could be 0.8 or greater. The assumption of no motion in this case would be justified and the values for F and N would be correct. But if the plane and the block are smoother and it is known that the coefficient of friction is only 0.4, then it can be shown that N is still 25 pounds but F would be limited to 0.4×25 pounds and would be 10 pounds. The block would then start from rest and, provided the force P remained constant in magnitude and direction, would acquire a velocity of 8.05 feet per second in the first second. In this case F would be assumed to be known to be $0.4N$, and the unknowns of the problem would be the normal force N and the acceleration of the body.

3. Fundamental Quantities. The first case discussed above involves position and direction. These are properties of *space*. The problem also involves the intensity of the pull of the cord, the weight of the body, and the forces exerted by the plane. These are *forces*. Every problem of statics involves the two fundamental ideas of force and space. When mass is involved in any problem of statics it is only because the mass is affected by gravitational forces which are significant.

The second case considered above is a problem of dynamics. In addition to the ideas of force and space, it involves the ideas of *mass* and *time*. The quantities of velocity and acceleration referred to are quantities involving space and time, velocity being length divided by time and acceleration being velocity divided by time.

The four fundamental quantities of mechanics are *force*, *mass*, *space*, and *time*. These quantities are elementary and the many attempts to define them have proved more wordy than useful. Every rational person, however, has a fairly definite notion of these quantities which he has acquired through his senses from the experience of his lifetime.

4. Standards and Units. While force, mass, space, and time are difficult to define, they are relatively easy to measure. It is necessary only to have for each quantity a *unit of measure*. If such measurements are to be taken at diverse times and places, it is necessary to have a *standard unit* to which all other units are referred. These standard units were originally chosen arbitrarily. Other units might have been selected just as well. When a particular unit has once been adopted, however, it is important that its value be preserved without change, in order that physical measurements separated by wide intervals of time may be compared with accuracy.

5. Length. Space in one direction is *length*. Two official standards of length are preserved by the Bureau of Standards at Washington. These are the standard yard, which is practically equal to the British standard yard, and the standard meter, which is a copy of the International Standard. The British standard yard is the "Straight line or distance (at 62°F) between the transverse lines in two gold plugs in a bronze bar deposited in the office of the Exchequer." The International Standard meter is the distance between two lines (at 0°C) on a platinum-iridium bar deposited at the International Bureau of Weights and Measures. The length of the standard meter in terms of the wave length of cadmium-vapor light has been carefully determined by Michelson. If this standard bar and the copies preserved by various nations should undergo any change in length, the magnitude of the variation may be found by a new comparison with the wave length of this light.

While the *yard* is the standard of length, the *foot* is the unit commonly used by American engineers. Physicists use the *centimeter*. In countries where the metric system has been adopted, the *meter* is the unit in engineering practice.

Space in one dimension is *length;* space in two dimensions is *area;* space in three dimensions is *volume*.[1]

The relation between the metric system and the inch is given by

$$39.370 \text{ in.} = 1 \text{ m}$$
$$2.540 \text{ cm} = 1 \text{ in.}$$

6. Time. The standard of time is the *mean solar day*. The unit commonly used in problems of mechanics which involve change of motion is the *mean solar second*. For other purposes longer units such as the *minute*, the *hour*, or the *day* are frequently employed.

7. Mass. The British standard of mass is the standard pound, which is a piece of platinum, marked "P.S. 1844, 1 lb" preserved at the Exchequer office. The metric standard of mass is the International kilogram, which is a piece of platinum-iridium alloy deposited at the International Bureau of Weights and Measures. Copies of the British standard pound and the International kilogram, preserved by the Bureau of Standards at Washington, are the official standards for the United States.

For engineering and commercial purposes, the *pound* is the common unit of mass in English-speaking countries. The *ton* is the unit for large masses. In Great Britain and in some of the states of the United States, the *long ton* of 2,240 pounds is official. Physicists and chemists use chiefly the *gram*. Units of mass are commonly called *weights*. A 10-pound weight as used on a beam balance is a 10-pound mass.

To convert from the metric system to the avoirdupois system, the relations are

$$15.432 \text{ gr} = 1 \text{ gram}$$
$$453.6 \text{ grams} = 1 \text{ lb}$$

8. Force. There is no *standard of force*. The common *unit of force* is the attraction of the earth for a unit of mass. A pound force is the force exerted by the earth on a pound mass. The force of attraction of the earth for a given body is called by physicists the *weight* of the body. A 10-pound mass weighs 10 pounds. Gravitational attraction between two spherical bodies varies directly as the product of the masses and inversely as the square of the distance between their centers. The *weight* of a body is directly proportional to its mass. If the earth were a homogeneous sphere and did not rotate on its axis, the weight of a body, as measured by a spring balance, would be the same at all points on its surface at equal distances above sea level. Since the earth is a spheroid with its polar radius about 13 miles shorter than its equatorial radius, the weight of a body, as measured by a spring balance, increases with the latitude.

---

[1] For a thorough discussion of standards and units, read the introduction of "Smithsonian Physical Tables."

Since the practical units are determined from the weights of standard units of mass, it is necessary to choose some standard location for the definition of these units of force. Sea level at 45° latitude has been suggested as the standard location. The standard pound force (which is not official) is defined as the weight of a pound mass at the standard location. A pound mass will weigh 0.997 pound at sea level at the equator and 1.003 pounds at the North Pole on a spring balance which is correct at the standard location. This variation is important in the determination of physical constants but is usually neglected in engineering calculations.

9. Work. Figure 2 shows a body B and a force **P** acting at a point C, which is called the *point of application*. This force may be exerted through a flexible cord which is tied to the body at the point of application. The body moves to a second position, which is represented by the dotted figure, while the force **P** continues to exert a constant pull in the same direction. The point C moves a distance s, which is called the *displacement* of the point of application. If the body does not rotate, the displacement of all parts is the same. If there is any rotation, the displacement of the point of application may be different from that of other points on the body.

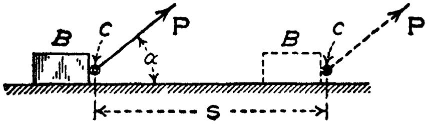


Fig. 2.

If the displacement of the point of application makes an angle α with the direction of the force, the work done by the force during the displacement is defined by the equation

$$\text{Work} = P \cos \alpha s \qquad \text{Formula I}$$

Since $P \cos \alpha$ is the component of the force in the direction of the displacement, the definition of work may be stated as follows: *The* *work of a force is the product of the displacement of its point of application multiplied by the component of the force in the direction of the displacement.*

Since $(P \cos \alpha) \times s = P \times (s \cos \alpha)$, work may be defined as the *product of the force multiplied by the component of the displacement in the direction of the force.*

If the force is expressed in pounds and the displacement in feet, the work is in *foot-pounds.* Other units of work are *inch-pounds, gram-centimeters, kilogram-meters,* and *ergs.*

Both force and displacement are *vectors,* hence work is the product of two vectors. There are two kinds of vector products. When the product of the magnitude of the two vectors is multiplied by the *sine* of the angle between them, the result is a vector and is called the *vector product.* When the product of the magnitude of the two vectors is multiplied by the *cosine* of the angle between them, the result has magnitude but no direction and is called the *scalar product.* Work is a scalar quantity without direction. Work may be positive or negative. Two or more quantities of work are added algebraically, with no attention to the direction of the forces and displacements by which the work was done.

Problems

1. A body is pulled 60 ft along a horizontal plane by a rope which makes an angle of 25° with the horizontal. The tension in the rope is 40 lb. Find the work done.
Ans. 2175 ft-lb.

2. A mass of 200 lb is pulled 40 ft up a plane which makes an angle of 15° with the horizontal by a rope which makes an angle of 45° with the horizontal. The tension in the rope is 80 lb. Find the work done by the rope.
Ans. 2771 ft-lb.

3. In Prob. 2, what is the vertical displacement of the mass? What is the work done on the mass by the attraction of the earth? What is the total work done on the mass by the pull of the rope and the attraction of the earth?
Ans. 10.353 ft; −2071 ft-lb; 700 ft-lb.

10. Measurement of Mass. The accurate measurement of mass is done by weighing on a beam balance. If a body A on the left pan balances a body B on the right pan, and if a body C on the left pan also balances the body B on the right pan, then the mass of A is equal to the mass of C. In other words, if the *weight* of A is equal to the *weight* of C, then the *mass* of A is equal to the *mass* of C. If the mass of C is 1 pound, the mass of A is 1 pound.

Weighing on a beam balance is a comparison of *forces.* Comparison of mass by weighing is based on the following definition: *The mass, or* *quantity of material, of a given body is proportional to the force with which it is attracted by the earth.*

Problems

1. If the radius of the earth is 4000 miles, what is the weight on a spring balance at 1000 miles above the earth of a body which weighs 10 lb at the surface? What is the mass at each position?
2. On a planet having a radius of 2000 miles and a mass one-tenth that of the earth, a 40-lb mass is weighed on a spring balance which reads in pounds at the earth's surface. What is the reading on the spring balance?
3. Calculate the length of a foot in centimeters.
4. Calculate the length of a meter in feet.
5. Find the mass of 1 kg in pounds, correct to four significant figures.

11. Representation of Quantity. There are several ways of representing magnitude. The most common is by means of numbers, as 6 feet, 8 pounds, 12 seconds. A number expresses magnitude in terms of a unit and means little to one who does not possess a definite idea of the magnitude of the unit. Two numbers give a clear notion of the *relative size* of two quantities without conveying any information as to the *actual size.* Anyone knows that 20 decameters is twice as long as 10 decameters, but if he is accustomed to measuring with English units, he must translate these lengths into yards or rods before he can have any real conception of their meaning.

While working with mechanics or any other science, the student must have an exact understanding of the physical nature of the units involved if his calculations are to be anything more than a meaningless substitution in formulas. Mechanics is a physical science in which mathematics is freely used as a tool in reasoning or as a shorthand method of expressing results. The mathematics is of small value, however, unless each step and every equation is interpreted in terms of the physical nature and the fundamental units of the quantities under consideration.

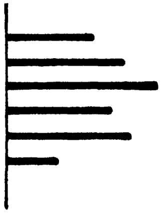


FIG. 3.

The *relative* magnitudes of several quantities are frequently represented by means of straight lines, as in Fig. 3. These lines may be horizontal, as in Fig. 3, or vertical with their lower ends on a horizontal line. For most purposes, such lines are merely used to express magnitude to the eye. The necessary calculations are made by numbers. The operations of addition and subtraction, however, may be conveniently performed with lines. In Fig. 4 it is desired to find the sum of the lines *ab* and *cd*. The lines are placed together so as to form one continuous line without overlapping. The total line thus formed is the sum of the two lines. It is immaterial in what order the lines are placed. The line *ad* of Fig. 4 is *ab* + *cd*. The line *cb* is *cd* + *ab*.

12. Vectors. Many quantities, such as mass, time, volume, or value, may be represented by a single number or by the length of a single line in any arbitrarily chosen direction. These are called *scalar* quantities. A quantity which has both magnitude and direction is called a *vector* quantity. A vector is expressed in words by giving its magnitude (length) and its direction. When the plane of the vector is known and a fixed directional axis is understood, a single angle is sufficient to designate the direction. For instance, a given vector is six units in length and makes an angle of 25° with the horizontal. To describe fully a vector in space, one magnitude and two angles are required. For instance, the magnitude of a vector is 10 units at an angle of 25° with the vertical in a vertical plane which is north 30° east.

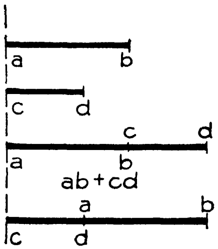


FIG. 4.

In Fig. 5, *ab*, *cd*, and *ef* are vectors in the plane of the paper. The vectors *ab* and *ef* have the same direction and equal length and, therefore, are equal.

In the vector *ab*, the point *a* is the *origin* and the point *b* is the *terminus*. The vector is regarded as extending from the origin to the terminus, as is indicated by the arrow. The arrow points from the origin toward the terminus, and its direction is the positive direction of the vector.

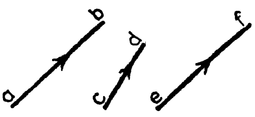


FIG. 5.

When a vector is designated by a single letter, that letter is usually printed in **boldface type.** In Fig. 6, **a** and **b** represent two such vectors. The arrow shows the origin, direction, and terminus.

In algebraic formulas when a single letter is used to express the magnitude of a vector while its direction is given in terms of angles, the letter is printed in *italic* instead of **boldface type.** In this book, a letter (such as **P** or **Q**) is often used to represent a force or other vector quantity. When emphasis is placed on both the magnitude and the direction of the quantity, these letters are printed in **boldface type.** When only the magnitude is stressed, *italics* are used.

13. Addition of Vectors. The addition of vectors is defined in the same way as the addition of lines (Art. 11). The origin of the second vector is placed at the terminus of the first vector and the line extending from the origin of the first vector to the terminus of the second vector is the vector sum. The addition of numbers (which may be represented by vectors in one direction) is a special case of vector addition.

Figure 6,II shows the addition of vectors **a** and **b** to get their sum **a** + **b**. Figure 6,III shows **b** + **a**. The vector **a** is drawn first in Fig. 6,II and the vector **b** is drawn first in Fig. 6,III. Figure 6,IV shows both additions in a single diagram, beginning at a common origin. Since the two lines **a** are equal and parallel and the two lines **b** are equal and parallel, the four lines form a parallelogram. The diagonal shown is the *vector sum*. Evidently it is immaterial in what order the vectors are added.

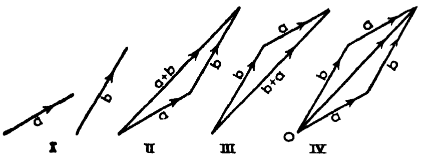


Fig. 6.

The sum of three or more vectors is found in the same way. Figure 7,II shows the addition of three vectors in the order **a** + **b** + **c**. Figure 7,I may be regarded as a graphical statement of the vectors to be added with each of the vectors starting from a common origin.

In Fig. 6,I, the graphical statement of the two vectors to be added, the origins of the vectors do not coincide. In Fig. 7,I, the three vectors to be added are shown with a common origin. In either case the addition is made in exactly the same way, and the vector sum is entirely independent of whether the vector quantities added are concurrent or not. Obviously, the sum of a set of vectors can be found *graphically* by drawing the vector diagram to scale, scaling the magnitude of the resultant, and measuring the angle which it makes with some convenient line. Or it can be found *trigonometrically* by sketching the diagram and computing the unknown quantities by the principles of trigonometry. The choice between the two methods depends on convenience and the degree of precision desired.

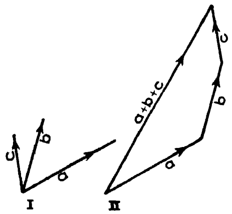


Fig. 7.

Problems

1. Given two vectors, **a** = 5 units at 0° and **b** = 4 units at 40°. Solve graphically for the vector sum **a** + **b**. Use the scale of 1 in. = 1 unit. Measure the vector sum. Measure the angle between the direction of the vector sum and that of vector **a**. Give the result in degrees.

First construct the statement to scale, as shown in Fig. 8,I. Then draw **a** as in Fig. 8,II equal and parallel to **a** of the statement. From the terminus of **a** draw **b** equal and parallel to **b** of the statement.

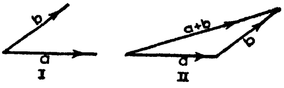


FIG. 8.

2. *Find the vector sum of three vectors in the same plane.* These are **a** = 12 units at 0°, **b** = 10 units at 30°, and **c** = 15 units at 75°. Find **a** + **b** + **c**. Use the scale of 1 in. = 5 units. On the same sheet draw also **c** + **a** + **b**. Measure the magnitude and direction of the vector sum on each drawing.
3. Find the vector sum of 20 units at 30° with the horizontal toward the right and 16 units at 15° with the horizontal toward the left.

14. Components of a Vector. The sum of two or more vectors is frequently called the *resultant* vector, and the vectors which are added are called *components* of the resultant. The process of finding the components is called *resolution.* In Fig. 8, the vector **a** + **b** is the resultant of the vectors **a** and **b**, and the vectors **a** and **b** are components of **a** + **b**.

Problems

1. A vector 20 units in length at 15° to the right of the vertical is made up of two components. One component is 60° to the right of the vertical and the other is 45° to the left of the vertical. Find the magnitude of these components.
2. Solve Prob. 1 if one component is 60° to the left of the vertical and the other is 45° to the right of the vertical.
3. A vector 15 units in length at 30° to the right of the vertical is the resultant of a vector 12 units in length and a vector 15° to the left of the vertical. Find the direction of one component and the magnitude of the other.

The most important kind of resolution of vectors is that in which each component is formed by the orthographic projection of the resultant upon a line along the desired direction. In Fig. 9, **c** is a vector which makes an angle θ with the horizontal. Its horizontal component is **a** and its vertical component is **b**. The lengths of these components are given by the equations

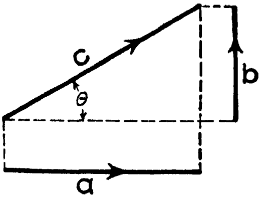


FIG. 9.

$$a = c \cos \theta$$
$$b = c \sin \theta$$

(In these equations, the letters a, b, and c represent magnitudes only. For that reason they are printed in *italics* instead of in **boldface type.** The components are vectors.)

Figure 10 shows two such orthographic components, which with their resultant form a right triangle.

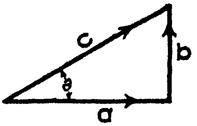


FIG. 10.

When the term *component* is used without qualification, the orthographic component is generally meant. The process of finding the orthographic component of a vector in a given direction is called *resolution* in that direction.

Problems

4. A vector of 60 units is north of 35° east. Find its component north and its component east. *Ans.* 49.149 units north; 34.415 units east.
5. A vector of 32.427 units is directed north 32°24′ east. Find its north component and its east component by logarithmic functions.
6. The horizontal component of a vector is 18.24 ft and the vertical component is 12.48 ft. By means of the tangent find the angle which the vector makes with the horizontal. Then find the length of the resultant vector by means of the cosine (or secant) of this angle. Check by projecting the horizontal and vertical components upon the line of the resultant and adding the components thus found (Fig. 11).
7. The horizontal component of a vector is 27.734 ft. The vertical component is 18.245 ft. Find the direction of the resultant vector by means of the logarithmic tangent. Find the magnitude of the resultant by means of the logarithmic cosine. Check by projections as in Prob. 6.

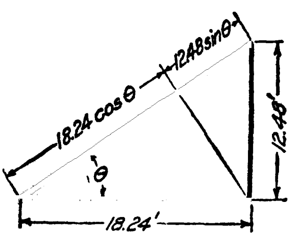
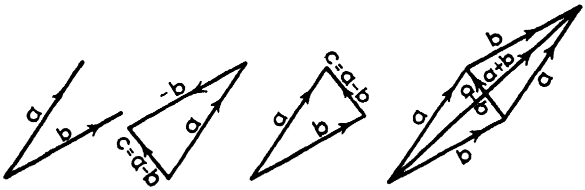


FIG. 11.

15. Vector Subtraction. Subtraction is the addition of a negative quantity. If a negative quantity is regarded as having direction, its direction is opposite to that of a positive quantity. *The negative sign before a vector reverses its direction.* If a vector **a** = 4 units at 25°, −**a** = 4 units at 205°.



I II III IV

FIG. 12.

Figure 12,I shows two vectors **a** and **b**. To get the difference **a** − **b**, the vector **a** is first drawn in Fig. 12,II. From the terminus of **a** the vector $-\mathbf{b}$ is then drawn in the direction opposite to that of **b** in Fig. 12,I. The vector **c** from the origin of **a** to the terminus of $-\mathbf{b}$ is the required difference.

$$\mathbf{a} - \mathbf{b} = \mathbf{c}$$
$$\mathbf{a} = \mathbf{c} + \mathbf{b}$$

Figure 12,III shows another method of getting the same result. If $\mathbf{c} = \mathbf{a} - \mathbf{b}$, then $\mathbf{a} = \mathbf{b} + \mathbf{c}$. The difference **c** is the vector which must be added to vector **b** to give vector **a**. The vectors **a** and **b** are drawn from the same origin and the terminus of **b** is then joined to the terminus of **a**. The vector **c**, with its origin at the terminus of **b** and its terminus at the terminus of **a**, is the required difference.

Figure 12,IV shows this form of vector subtraction combined with vector addition. The vectors **a** and **b** form the sides of a parallelogram. The diagonal which starts at the origin of the first two vectors is the vector sum. The other diagonal is the vector difference.

Problems

(Solve graphically)

1. Given a vector of 12 units at 0° and a vector of 20 units at 30°. Subtract the first vector from the second *Ans.* 11.33 units at 62°.
2. Vector $\mathbf{a} = 15$ units at 20°, vector $\mathbf{b} = 12.5$ units at 40°, and vector $\mathbf{c} = 20$ units at 110°. Find $\mathbf{a} + \mathbf{b} - \mathbf{c}$. Find $\mathbf{a} - \mathbf{c} + \mathbf{b}$. Find $\mathbf{c} - \mathbf{a} - \mathbf{b}$.

16. Vector Multiplication. In mechanics, *work* is defined as *force multiplied by displacement*, it being understood that the force is in the same direction as the displacement. Force and displacement are both vector quantities, and if the two vectors are not in the same direction, no work is done by the component of the force perpendicular to the direction of the displacement, and the quantity of work is found by multiplying the component of force in the direction of the displacement by the magnitude of the displacement. The product, *work*, is a scalar quantity and is called the *scalar product* of the two vectors. In Fig. 13, if the force P remains constant in magnitude and direction while it pulls the body B a distance S, work $= P \cos \alpha S$. *The scalar*

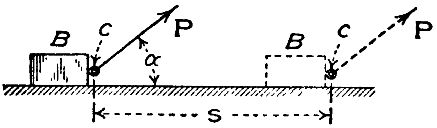


FIG. 13.

product of two vectors is the product of the magnitudes of the two vectors multiplied by the cosine of the angle between their directions.

Moment, as the term is used in mechanics, is a measure of the tendency to turn. In Fig. 14, the force **Q** in the inclined cord tends to turn the bar in a clockwise sense about the hinge at the lower end of the bar. This moment equals the magnitude of the force **Q** multiplied by the perpendicular distance from the hinge to the cord, or the magnitude of **Q** multiplied by the length of the bar and by the sine of the angle between the cord and the bar. Moment is the *vector product* of two vectors, a force and a length. *The vector product of two vectors is the product of the magnitudes of the two vectors multiplied by the sine of the angle between their directions.*

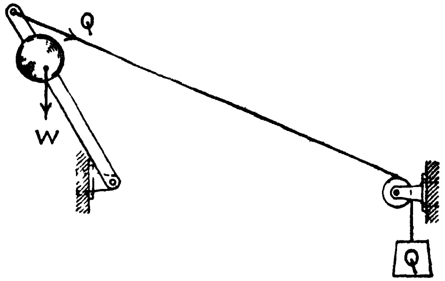


FIG 14.

In works using *vector notation*, the vector quantities are usually indicated by **boldface type.** The vector product is called the *cross product* and is indicated by an ordinary multiplication sign. In the example above, calling the moment M, the length of the bar L, and the angle between the cord and the bar α, we could write $\mathbf{M} = (QL) \sin \alpha = \mathbf{Q} \times \mathbf{L}$. In this notation, the scalar product of two vectors is called the *dot product*, the multiplication being indicated by a dot between the quantities multiplied. Let U be the work in the example above, and the expression for the work can be written as follows: $U = (PS) \cos \alpha = \mathbf{P} \cdot \mathbf{S}$.

A vector multiplied by a real number n is simply a vector in the same direction as the original vector but n times as great. It is for this reason that a *unit vector* of any chosen magnitude can be set up in any direction and other vectors in the same direction can be designated by numbers which indicate the magnitude of the vectors in question in terms of the unit originally chosen. A scalar multiplied by a real number n is a scalar quantity of the same kind as the original scalar but n times as great; so, similarly, a designated unit of any scalar quantity can be used to indicate larger or smaller quantities, such as 4 *pounds, half a bushel.* A vector multiplied by a scalar is a vector. A scalar multiplied by a scalar is an absurdity.

Of the fundamental quantities of mechanics, two (space and force) are vector, and two (mass and time) are scalar, so that the concept of the vector is an absolute essential to the understanding of mechanics. Fortunately, an intimate knowledge of vector analysis is not at all necessary to the understanding of the principles of mechanics and the art of applying them to practical problems. The simple elements stated above are sufficient for most of the useful applications. With few (and questionable) exceptions, sums and products of quantities having both magnitude and direction are correctly determined by vector addition and multiplication.

17. Dimensional Equations. A dimensional equation is an equation which shows what exponents should appear with the fundamental quantities in expressions for physical quantities. In mechanics, mass, length, and time, represented by M, L, and T, are usually taken as the basic quantities. Negative exponents are generally used instead of fractional forms and the physical quantities are defined, for example, as

Area $= L^2$
Volume $= L^3$
Density $= ML^{-3}$
Force $= MLT^{-2}$
Displacement $= L$
Velocity $= LT^{-1}$
Acceleration $= LT^{-2}$
Moment $= ML^2T^{-2}$

In dealing with problems of statics, it is much simpler to use force as a fundamental quantity (since mass and length do not appear in that restricted branch of mechanics). Some authors prefer to use force as a fundamental quantity and mass as a derived quantity. Mass then equals $FL^{-1}T^2$.

Dimensional equations have three important uses:

1. A dimensional equation may be used to determine whether a literal expression or equation is possibly correct.

2. In transferring from one system of units to another, a dimensional equation indicates the required power of the ratio of the units of the two systems.

3. In seeking to formulate a mathematical expression for a physical law, a study of the dimensions involved may indicate the form of the expression sought and serve as a guide to a logical experimental procedure.

The first two of these uses are of interest to the student of this text. It will normally suffice to put into all terms the actual units used and then inspect the equation to be sure that the units could be canceled as though they were numbers, or that "the dimensions check." For example,

$$\frac{(6 \text{ ft per sec})(12 \text{ lb})}{32.2 \text{ ft per sec per sec}} = (80 \text{ lb})(t \text{ sec})$$

An inspection shows that the units check. This is the equation of impulse and momentum, mass × velocity = force × time. If the equation had been written erroneously, omitting the 32.2 which is the acceleration of gravity, g, then the equation would have been dimensionally inconsistent.

The dimensions of an area are $A = L^2$. If an area in square yards is transferred to square feet, the ratio of the number of units of length in feet to the number of units in yards is 3. Since L in the dimensional equation is squared, it follows that the number of square feet in a given area is 3^2 times as great as the number of square yards. This is an example of the second use of a dimensional equation.

If a, b, and c are lengths, an area may be expressed by a^2, b^2, c^2, ab, ac, bc, $c\sqrt{a^2+b^2}$, etc. If an expression which is known to be an area has any of these forms, or any other form which is equivalent to the square of a length, the expression represents an area. Such expressions as abc, $(a^2+b^2)/c$, $\sqrt{abc}$, etc., cannot represent areas. If such an expression appears in a quantity which is known to be an area, it will be recognized as incorrect. These are examples of the first use of a dimensional equation.

Only quantities having the same dimensions may be added or subtracted. Volume cannot be added to area nor velocity to density. If a, b, and c are lengths and m is a mass, $ab + b^2 + a\sqrt{bc}$ and $mc + m\sqrt{a^2+c^2}$ are possible additions.

Problems

1. The dimension of the trigonometric function is zero, since each function is the ratio of two lengths. If a, b, c, and s are lengths, is it possible to represent a sine by $\frac{b}{\sqrt{a^2+c^2}}$, by $\frac{ab}{c^2}$, by $\sqrt{\frac{(s-b)(s-c)}{bc}}$, or by $\frac{s^2+bc}{a}$? What is the nature of the quantity expressed by $\sqrt{s(s-a)(s-b)(s-c)}$? What is the quantity expressed by $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$?

2. What is the coefficient required to reduce area in square feet to area in square inches and to area in square yards? *Ans.* 12^2, $(\frac{1}{3})^2$

3. What is the factor required to reduce pounds per square inch to grams per square centimeter?

4. How is velocity in feet per second reduced to velocity in miles per hour?

5. How is acceleration in feet per minute per minute reduced to acceleration in feet per second per second?

6. Show that the expression for radius of curvature on page 278 represents a length.

CHAPTER 2

CONCURRENT, COPLANAR FORCES

18. Resultant. When a system of forces is replaced by another system having the smallest possible number of forces without making any change in the effect of the system, this simpler system is called the *resultant* of the original system. The resultant of the most complicated force system can be represented by two vectors, and the resultants of the typical force systems usually found in engineering problems can be represented by one vector each. By definition of resultant, any system of forces can be replaced, if convenient in the solution of any problem, by its resultant or by any other system having the same resultant; conversely, any force can be replaced by a system of which it is the resultant.

The importance of the concept of the resultant and its usefulness in the solution of problems justify the separate consideration of the resultant for each class of force systems.

If the magnitude and direction of each of two concurrent forces are represented by a vector the direction and magnitude of the resultant of these forces are represented by the vector sum of these vectors. This is the *force-triangle law* (in a slightly different form called the *law of the parallelogram of forces*), which may be regarded as an axiom amply verified by experiment.

If the resultant is to have *exactly the same effect* as the two forces which it replaces, *it must be applied at the same point to which the forces it replaces were applied, i.e.*, at the point of concurrency. This, also, can be considered axiomatic. For the purpose of determining whether or not a system of forces is in equilibrium, any force or resultant may be considered to act anywhere along its line of action, on or off the free body. For example, if a body is suspended by a string, it makes no difference whether the string is attached to a support a short distance above the body or a greater distance if the question is simply whether the body is in equilibrium. But if the body is swinging as a pendulum or if the question is whether the body will return to the position of equilibrium if displaced and if so how quickly, the point of attachment of the string makes a great deal of difference. This distinction must be remembered in connection with the axiom: *A force may be considered as applied anywhere on its own line of action.*

Figure 15 illustrates two concurrent forces which are applied to a beam at a point *A* by means of two flexible cords. Each cord runs over a smooth pulley. The left cord supports a mass of 8 pounds. Since the pulley is smooth and the cord is flexible, it is known that the force exerted by this cord at *A* is 8 pounds tension directed along the axis of the cord. A measurement of the angle between the cord and a plumb line shows that the tension makes an angle of 35° with the vertical.[1] The force in the right cord is 12 pounds at an angle of 50° to the right of the vertical.

Figure 15,II is the *space diagram* giving accurately the angles of the two forces which are concurrent. The arrows show the direction of each force along its line. Figure 15,III is the *force polygon*, which, in this case, is a *triangle*. To draw the force triangle, a point *O* is selected as the origin and a line is drawn through this point parallel to the direction of the 12 pounds of the space diagram. A length of 12 units of some convenient scale is measured off from *O* to *C*. From *C*, a second vector, 8 units in length, is drawn parallel to the force of 8 pounds of the space diagram. The direction and magnitude of the resultant of these two forces are given by the line *OD* which is drawn from the origin of the first vector to the terminus of the second vector. The arrow on this resultant is from *O* toward *D*. Finally, through the point of application of the forces on the space diagram, a broken line is drawn parallel to the direction of the line *OD* of the force triangle. This broken line gives the location of the line of action of the resultant force.

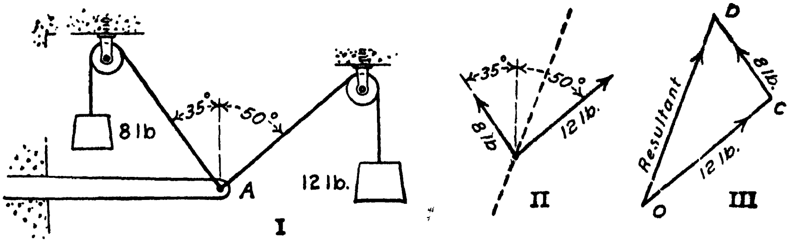


FIG. 15.

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[1] In this book when a direction is given by means of an angle with the vertical, the *vertical upward* is meant, unless otherwise specified. When a direction is given by means of the angle with the horizontal, the *horizontal* to the *right* is meant, and the positive angle is measured counterclockwise from the reference line.

Where a drawing of the structure is available on the sheet used for calculating the forces, as Fig. 15,I, this drawing may be used as the space diagram and Fig. 15,II may be omitted.

Problems

1. Find the magnitude and direction of the resultant of the following forces in a horizontal plane: 15 lb north[1] 75° east, 18 lb north 30° east, and 12 lb north 60° west. Solve graphically to the scale of 1 in. = 5 lb. Measure the length of the resultant in inches and reduce to pounds. Find the angle which the resultant makes with the north-south line by means of its sine using a 5-in. radius. Check this angle by means of its tangent.
2. Two ropes in a vertical plane are attached to a fixed point *A*. One rope, which supports a mass of 78 lb, runs over a smooth pulley which is 12 ft higher than *A* and 5 ft to the left of the vertical line through it. The second rope, which supports a mass of 50 lb, runs over a smooth pulley which is 12 ft higher than *A* and 9 ft to the right of the vertical line through it. Construct a space diagram to the scale of 1 in. = 5 ft. Then construct the force triangle to the scale of 1 in. = 30 lb. Measure the resultant and express its magnitude in pounds and its angle with the vertical in degrees and minutes.
3. Solve Prob. 2 with 50 lb on the left rope and 78 lb on the right rope.

19. Calculation of the Resultant of Two Forces. Since the force polygon for two forces and their resultant is a triangle, the resultant of two forces may be calculated by means of the formulas for the solution of triangles. If the two forces **P** and **Q** of Fig. 16 are at right angles to each other or if the force **P** is equal to the force **Q,** the magnitude and direction of the resultant may be computed by formulas for right triangles. For most cases the formulas for oblique triangles are required. The angle between the forces **P** and **Q** (Fig. 16,I) is the supplement of the angle opposite *R* in Fig. 16,II. From the cosine law

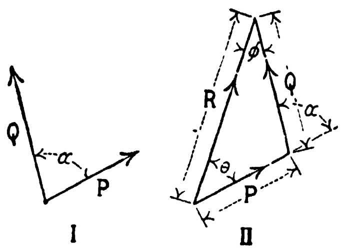

FIG. 16.

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha \qquad (1)$$

The angle θ between R and P and the angle ϕ between R and Q may be found by the sine law.

Problems

1. Look up the cosine law and compare with Eq. (1). Explain the difference in sign of the last term.

---

[1] Surveyors express directions by angles with the meridian. North 30° east is a direction of 30° east of north. South 20° east is 20° east of south.

2. Find the magnitude and direction of 240 lb at 50° with the horizontal and 160 lb at 30° with the vertical. *Ans.* $R = 330$ lb at 77°6′ with the horizontal.
3. Find the resultant of 60 lb at 35° with the vertical and 60 lb vertically upward. Also solve graphically to a scale of 1 in. = 10 lb and compare results. What is your error in pounds and what length on your scale does this represent? *Ans.* 106.44 lb at 62°30′ with the horizontal.
4. Solve graphically for the resultant of the four forces in the two problems immediately above. Check by adding the answers vectorially.
5. Solve Prob. 1, Art. 18, trigonometrically.

20. Calculation of the Resultant of More than Two Forces. Problem 4 of Art. 19 is an example of adding more than two forces. If the work is to be done graphically, all the vectors are laid off, in any order, with the origin of each at the terminus of the preceding one, and the vector from the original origin to the final terminus represents the sum. Each additional force in the problem adds to the difficulty of the solution only the labor required to lay off a single vector. If, however, the work is to be done trigonometrically, each additional force adds the labor of computing one additional triangle, and it is less labor to make use of the principle that the sum of the components of any system of forces in any given direction is equal to the component of the resultant in the same direction. Figure 17,I shows three forces, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, which make angles of α, β, and γ, respectively, with the axis OX. Figure 17,II shows the vector addition of these forces, together with components of each force along OX and along an axis OY perpendicular to it. The resultant of these forces is a line from O to C, which is not drawn on Fig. 17,II. The component of this resultant parallel to OX is $\mathbf{H}$, which is equal to the sum of the components of the forces.

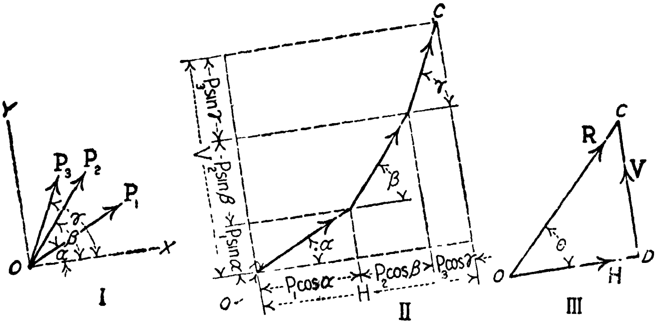

FIG. 17.

$$H = P_1 \cos \alpha + P_2 \cos \beta + P_3 \cos \gamma \tag{1}$$

The component of the resultant parallel to OY is **V**.

$$V = P_1 \sin \alpha + P_2 \sin \beta + P_3 \sin \gamma \qquad (2)$$

The components **H** and **V**, drawn with reduced scale, with their resultant **R** form the right-angled force triangle of Fig. 17,III. If θ is the angle which the resultant makes with the axis OX,

$$\tan \theta = \frac{V}{H} \qquad (3)$$

$$R = H \sec \theta = \frac{H}{\cos \theta} \qquad (4)$$

$$R = \frac{V}{\sin \theta} \qquad (5)$$

The resultant **R** might be calculated from the square root of the sum of the squares of **H** and **V**. If these numbers are of four or more figures, it is easier to use the equations above.

Example

Find the resultant of 10 lb at an angle of 20° with the horizontal, 12 lb at 60°, and 8 lb at 110°.

First Solution: Axes at 0° and 90°

The statement of the example is given by Fig. 18,I. (This figure also illustrates the construction of a 20° angle by means of its chord.) It is assumed that the forces are measured accurately to one-hundredth of a pound. Resolution along the axis OX gives

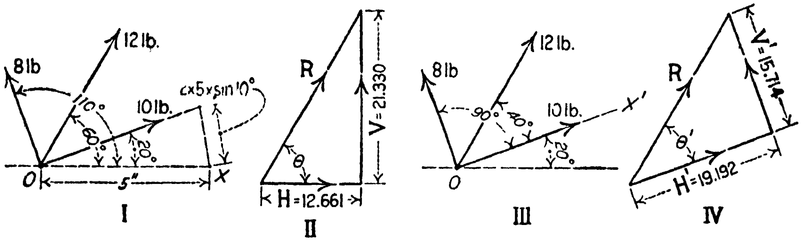

FIG. 18

$$\begin{aligned} 10 \cos\ 20^\circ &= 10 \times 0.9397 &&= 9.397 \\ 12 \cos\ 60^\circ &= 12 \times 0.5000 &&= 6.000 \\ 8 \cos 110^\circ &= 8 \times (-0.3420) &&= -2.736 \\ & & H &= 12.661 \end{aligned}$$

Resolution along the axis OY gives

$$\begin{aligned} 10 \sin\ 20^\circ &= 10 \times 0.3420 &&= 3.420 \\ 12 \sin\ 60^\circ &= 12 \times 0.8660 &&= 10.392 \\ 8 \sin 110^\circ &= 8 \times 0.9397 &&= 7.518 \\ & & V &= 21.330 \end{aligned}$$

From Fig. 18,II,

$$\tan\theta = \frac{21.330}{12.661} = 1.6847$$

$$\theta = 59°18'$$

$$R = \frac{21.330}{\sin 59°18'} = \frac{21.330}{0.8599} = 24.81 \text{ lb}$$

The student will check R by means of H and the cosine of θ.

Second Solution: Axes at 20° and 110°

The statement[1] of the example is given by Fig. 18,III.

The solution is arranged in tabular form, which is very convenient when the number of forces is large.

Force	*Angle*	*Cosine*	*H′ component*	*Sine*	*V′ component*
10	0°	1	10.000	0	0
12	40°	0.7660	9.192	0.6428	7.714
8	90°	0	0	1	8.000
			$H' = 19.192$		$V' = 15.714$

21. Equilibrium. The vector diagram, Fig. 15,III, gives the resultant of the 8-pound force and the 12-pound force acting at point A. If the two forces were removed and a single force equal to this resultant were applied at point A, the effect on the beam would be unchanged. But point A does not move because the beam applies at point A to the cords a force exactly equal and opposite to the resultant of the other two forces. The three forces are in *equilibrium.* The point of attachment A is also said to be in equilibrium, and if there is a small body at A, such as a nail to which the cords are attached, this small body is said to be in equilibrium. The *resultant of all three forces* is zero and their vector sum is zero, as can be seen by reversing the resultant in the force diagram at the right of the figure. *Any body which remains stationary or which moves in a straight line without change of speed is in equilibrium.*

Newton's *first* law states that: *Every particle continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change that state.* This law follows rationally from the second law, to be discussed later.[2] The law means, simply, that so long as no vector quantity results from all the forces acting on the particle, there will be no change in its state of motion. It can be easily shown that the law thus stated applies to any sizable body as well as to a particle, it being remembered that "vector quantity" includes moments as well as forces.

---

[1] It is not necessary to make this sketch. Figure 18,I will do for both solutions.

[2] It is occasionally argued that since statics is but a special case of dynamics, its study should be deferred until after consideration has been given to the more general cases. However, a better mastery of any subject can usually be attained by starting with the simpler ideas and building upon thorough comprehension of them, and the usefulness of statics to the engineer transcends that of any other science so greatly that its thorough mastery must be conceded first priority.

When a body is in equilibrium, the forces acting on it are said to be *in balance.* Every force in the system is equal and opposite to the resultant of all the other forces in the system and acts along the same line as does this resultant. Further, the resultant of any group of the set of forces is exactly equal and opposite (along the same line) to the resultant of all the remaining forces. Any force in the system can be properly called the *equilibrant* of all the other forces, and the resultant of any group of the forces is the equilibrant of the remainder of the forces of the system. If there are only two forces in the system in equilibrium, the two forces are equal, opposite, and along the same line Each is the equilibrant of the other

Figure 19 shows three flexible cords attached to a point. The resultant of the forces **P** and **Q** is the force **R** of the vector addition diagram, Fig. 19,II. Figure 19,III is the force triangle for the three forces in equilibrium. For a graphical solution of this problem, Fig. 19,II would be unnecessary and would not be drawn. Figure 19,I is called the *space diagram* and is simply a sketch showing the relation of the forces in space and giving all the information known about the problem as well as indicating the unknowns. Figure 19,III is known as the *force diagram*, the *vector diagram*, or, if there are more than three forces involved, the *force polygon.* The positions of the lines in the vector diagram are not important and they may be laid off in any order, even with some of the lines crossing over others. It usually looks a little better, however, and is less confusing to lay them off in some order which gives an open polygon. The force diagram *should never be superimposed* upon the space diagram. Such a procedure never clarifies and often confuses the problem.

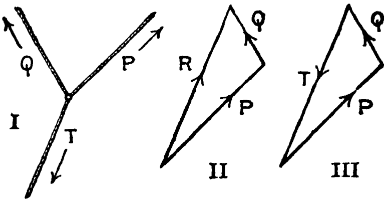

FIG. 19.

Figure 19 presents a basic problem of coplanar, concurrent, nonparallel forces. There are six elements, the magnitudes of P, T, and Q, and their respective directions (the point of concurrency and the plane in which the forces lie being understood). Figure 19,III is a solution of the problem. It has six elements, the lengths of the three sides and their respective directions, or, if preferred, the lengths of the sides, the direction of one side, and two of the angles of the triangle (the third angle of the triangle being known from the other two and a proposition of geometry). It should be clear from geometry that if two (and only two) of the six elements are not known, the force diagram can be constructed and the problem can be solved. For example, if all the directions in Fig. 19,I are known and the magnitude of **T** is given, there is a single definite value for each of the forces **P** and **Q.**

In the force diagram for a force system in equilibrium, the terminus of the last vector coincides with the origin of the first, and the arrow on one side of each angle of the diagram points toward the angle, while the arrow on the other side of the same angle points away from it. When a body is in equilibrium under the action of a set of forces, the force polygon closes and the arrows follow each other around.

Example

A 40-lb mass rests on a smooth inclined plane, which makes an angle of 35° with the horizontal. It is supported by a horizontal push of 12 lb, and a pull **P** parallel to the plane. Find the force **P** and the normal reaction.

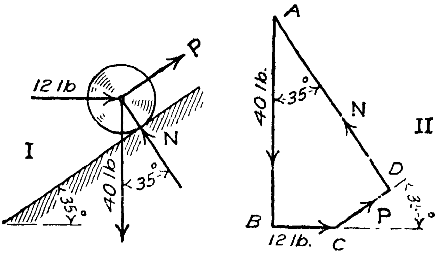

Fig. 20.

Figure 20,I is the space diagram showing the direction of all the forces. The mass of 40 lb is the *free body* in equilibrium. The arrows show the direction of the forces which act from other bodies on the free body. The reaction of the plane is upward at an angle of 35° with the vertical. The pull of the earth on the free body is downward force of 40 lb. The lengths of the lines in this diagram are not necessarily proportional to the magnitudes of the forces.

Figure 20,II is the force polygon. The known forces are 40 lb downward and 12 lb horizontally toward the right. The unknown quantities are the magnitude of the pull **P**, the direction of which is known to be parallel to the plane, and the magnitude of the normal reaction of the plane. As the first step in the construction of the force polygon, a vertical line *AB* is drawn which is 40 units in length. From *B* a horizontal line *BC* is drawn which is 12 units in length. The arrow in *AB* is downward toward *B* and the arrow in *BC* is toward the right, directed away from *B*. Through *C* a line is drawn parallel to the force **P** of the space diagram, and through *A* a line is drawn parallel to the direction of the normal reaction. These lines are extended until they intersect at *D*. The length of *CD* measures the tension parallel to the inclined plane, and the length of DA represents the normal reaction of the plane against the 40-lb mass. Since the arrow in BC is toward C, the arrow in CD must be away from C toward D. The arrow in DA is from D toward A, which is the *point of beginning* of the polygon.

As Fig. 20,II is drawn, the forces of the space diagram follow each other in the counterclockwise direction. The force of 12 lb might have been placed at the top to the left of A. The forces could then follow in the clockwise direction on the space diagram, and the force polygon would be inverted. Other forms of the polygon are possible if the forces are not taken in order around the point of application. The results are the same for any possible arrangement.

Problems

1. Solve the above example graphically to the scale of 1 in. = 10 lb, with the horizontal force 17.5 instead of 12 lb.
2. Solve Prob. 1 if the horizontal force is 35 lb.
3. A 50-lb mass is supported by three cords in the same vertical plane. One cord makes an angle of 45° to the left of the vertical. The second cord makes an angle of 30° to the right of the vertical. The third cord runs horizontally toward the right and exerts a pull of 10 lb. Find the tension in each of the others, using the scale of 1 in. = 10 lb.
4. A mass of 50 lb is supported by two ropes; one makes an angle with the vertical whose tangent is 0.75 and the other exerts a pull of 45 lb. Find the tension in the first rope and the direction of the second. Measure the angle by means of the tangent. Give the results for both possible solutions.
5. Solve Prob. 4 if the second rope exerts a pull of 60 lb.

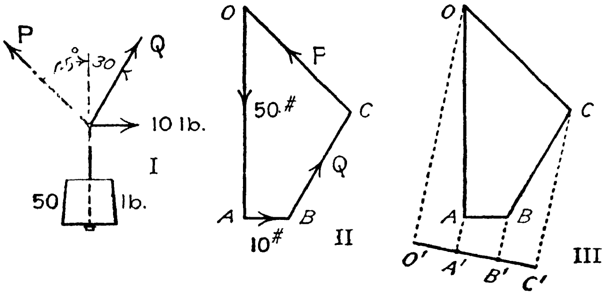

FIG. 21.

22. Equilibrium by Resolutions. The force diagram for concurrent, coplanar forces in equilibrium is a closed polygon. If a closed polygon is projected upon any line in its plane, the sum of the positive projections is equal to the sum of the negative projections, and the total projection is zero. When the forces which act on a body are in equilibrium, their resultant is zero and the sum of their components in any direction is zero.

Figure 21,I is the space diagram for Prob. 3 of Art. 21, and $OABCO$ of Fig. 21,II is the force polygon. Figure 21,III shows the projections of the sides of the force polygon upon a line at an angle of 10° below the horizontal. The projections $O'A'$, $A'B'$, and $B'C'$ extend from the *left* toward the *right*. The projection $C'O'$ extends from the right toward the left to the point of beginning.

To solve a problem of equilibrium of concurrent forces by resolutions, it is not necessary to draw the force polygon, as the components may be calculated from the space diagram. However, *always draw the space diagram and mark all the forces which act on the free body.*

Figure 22 shows the space diagram for four forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$, which make angles α_1, α_2, α_3, and α_4, respectively, with the horizontal line toward the right. The point at which the forces meet is the free body. The components along the horizontal are $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, etc. If the forces are in equilibrium,

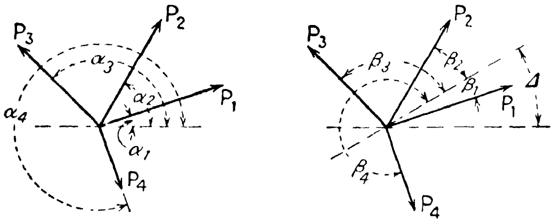

FIG. 22.

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4 = 0 \qquad (1)$$

If the forces of Fig. 22 make angles β_1, β_2, β_3, and β_4 with a second axis, the second condition of equilibrium is

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + P_4 \cos \beta_4 = 0 \qquad (2)$$

For Eq. (1), the dotted horizontal line in Fig. 22 is called the *axis of resolution*, or simply the *axis*. The term $P_1 \cos \alpha_1$ is called the *component of the force* $\mathbf{P}_1$ *on the axis.* A resolution equation is written by multiplying each force acting on the body by the cosine of the angle which it makes with a chosen axis and setting the sum equal to zero. For any system of forces in equilibrium in a plane, this operation can be done twice independently. If the operation is done a third time, the equation so written is true but not independent. These ideas are written in mathematical cryptology thus:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \text{Formula II}$$

For Eq. (2), the dotted line making the angle Δ with the horizontal is the axis, and the equation is the result of multiplying each force by the cosine of the angle it makes with this axis and setting the sum equal to zero. Three things should be noted here about the practical business of writing resolution equations.

1. *Any axis may be used* and the resulting equation will be correct unless a mistake occurs. The student who slavishly adheres to the practice of resolving only on vertical and horizontal axes will miss most of the opportunities for skillful and elegant solutions.

2. "Multiplying each force by the cosine of the angle it makes with the axis" presupposes that one direction along the axis has been taken as the positive direction and all angles are measured counterclockwise (clockwise would do as well) with this direction. This procedure involves using angles up to 360° and keeping careful track of the algebraic signs of the cosines lying in the several quadrants. For most students, it is easier and safer to multiply each force by the cosine of the acute angle it makes with the axis, and to determine the algebraic sign of the term by noting which way the arrow would point if the force were projected on the axis.

3. It is not necessary to draw a separate space diagram for each resolution as is shown in Fig. 22. It is not even necessary or desirable to show a diagram with all forces radiating outward from the center of concurrency as is done in Fig. 22. It may readily happen that the free-body diagram may be clearer if one or more forces are shown acting toward the center of concurrency, as is N in Fig. 20. The direction of the component of this force on any axis will not be made any clearer by replacing N by a string pulling upward 35° to the left of the vertical.

Any number of equations may be written by changing the direction of the axis of resolution. Since two components completely determine a vector in a given plane, only two such equations can be independent. A third equation may be written as a check.

Example 1

A mass of 80 lb is supported by three flexible cords in the same vertical plane. One cord makes an angle of 30° to the left of the vertical and exerts a pull of P lb. The second cord makes an angle to the right of the vertical whose cosine is 0.6 and exerts a pull of Q lb. The third cord makes an angle of 75° to the right of the vertical and exerts a pull of 12 lb. Find the unknown forces P and Q by horizontal and vertical resolutions and check by a third resolution.

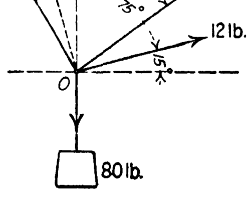

FIG. 23.

From the horizontal resolution (Fig. 23) with components toward the right as positive, the equation is

$$12 \cos 15° + Q \sin \beta - P \cos 60° = 0 \tag{1}$$
$$12 \times 0.9659 + 0.8Q - 0.5P = 0 \tag{2}$$

From the vertical resolution, positive upward,

$$P \cos 30° + Q \cos \beta + 12 \sin 15° - 80 = 0 \tag{3}$$
$$0.8660P + 0.6Q = 80 - 12 \times 0.2588 = 80 - 3.1056 = 76.8944 \tag{4}$$

From Eqs. (2) and (4),

$$\begin{aligned} 1.5P - 2.4Q &= 34.7724 & (5)\\ 3.464P + 2.4Q &= 307.5776 & (6)\\ 4.964P &= 342.3500 \text{ lb} & (7)\\ P &= 68.966 \text{ lb} & \\ 0.8Q &= 34.483 - 11.591 = 22.892 \text{ lb} & (8)\\ Q &= 28.615 \text{ lb} & \end{aligned}$$

The solution is now finished, but there may be errors in writing the equations or in the numerical work. The *check* is fully as important as the solution. A check may be made by resolution along any axis except those used for Eqs. (1) and (3). (A resolution perpendicular to Q checks only one unknown.) A resolution perpendicular to the known force of 12 lb (along the axis ON of Fig. 23 at an angle of 15° with the vertical) may be chosen. This gives

$$P \cos 15° + Q \sin 21°52' - 80 \cos 15° = 0 \tag{9}$$
$$P + \frac{28.615 \times 0.3725}{0.9659} = 80 \tag{10}$$
$$68.966 + 11.035 = 80.001 = 80$$

The error is 1 part in 80,000, which is much less than might be expected with four-place tables.

Problems

(*Draw the space diagram for each problem. Mark each force which acts on the free body.*)

1. A mass of 200 lb is supported by two ropes, one of which makes an angle of 60° to the left of the vertical and the other an angle of 45° to the right of the vertical. Solve by horizontal and vertical resolutions. Check by a resolution along an axis at 60° to the left of the vertical.

2. A horizontal rod AB, 8 ft in length, is hinged at the right end B (Fig. 24) and supported by a rope which is attached at the left end A. The other end of the rope is fixed to a point C, which is 6 ft above the hinge B. The rod supports a mass of 60 lb at A. Find the tension in the rope and the compression in the rod by vertical and horizontal resolutions. Check by a resolution parallel to the rope.

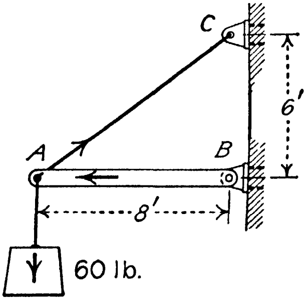

FIG. 24.

When a force is unknown, both as to direction and magnitude, it is best to consider this force as made up of two components at right angles to each other and find the magnitude of each of these components separately. Next find the direction and magnitude of their resultant, which is the force required. This method is the same as the one for finding the resultant of a set of concurrent forces.

Example 2

A 50-lb mass is supported by two cords, one of which makes an angle of 75° to the right of the vertical and exerts a pull of 30 lb. Find the direction of the second cord and the tension which it exerts.

In Fig. 25, the unknown force Q at an unknown angle θ with the vertical may be regarded as made up of a horizontal component **H** and a vertical component **V**. Resolving horizontally,

$$H = 30 \cos 15° = 30 \times 0.9659 = 28.977$$

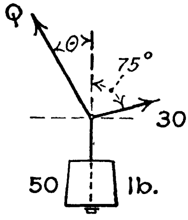

FIG. 25.

Resolving vertically,

$$V = 50 - 30 \sin 15° = 50 - 30 \times 0.2588 = 50 - 7.764 = 42.236$$

$$\tan \theta = \frac{28.977}{42.236} = 0.6861$$

$$\theta = 34°27'$$

$$Q = \frac{42.236}{\cos \theta} = \frac{42.236}{0.8246} = 51.22$$

Check by resolving parallel to Q,

$$50 \cos 34°27' + 30 \sin 19°27' = Q$$

$$50 \times 0.8246 = 41.23$$

$$30 \times 0.3330 = 9.99$$

$$51.22 = Q$$

Problems

3. A mass of 120 lb is supported by three cords in the same vertical plane. One cord exerts a pull of 60 lb in a horizontal direction toward the left. The second cord makes an angle of 60° to the right of the vertical and exerts a pull of 40 lb. Find the direction of the third cord and the tension which is exerts. Check.
Ans. 103.17 lb at 14°14′ to the right of the vertical.

A mass of 40 lb is supported by two cords. One is horizontal and the other exerts a force of 60 lb. Find the force of the first cord and the direction of the second. *Ans.* $H = 44.72$ lb; $\theta = 48°12'38''$ with the vertical.

23. Elimination of One Unknown. The examples in the preceding article were solved by horizontal and vertical resolutions, and the problems can be easily solved in the same way. But in writing a resolution equation *any axis* may be used, and it is often advantageous to eliminate an unknown by a skillful choice of an axis rather than by the solution of simultaneous equations. The following example is solved twice to indicate the amount of labor that may be saved if equations containing only one unknown are written.

Example

A 40-lb mass is supported by three flexible cords in the same vertical plane. One cord makes an angle of 35° to the left of the vertical and exerts an unknown pull of P lb. A second cord makes an angle of 25° to the right of the vertical and exerts an unknown pull of Q lb. The third cord makes an angle of 80° to the right of the vertical and exerts a pull of 10 lb. Find the unknown forces by resolutions.

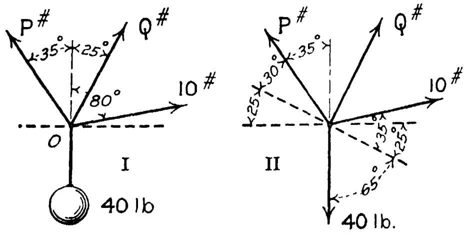

FIG. 26.

The point O (Fig. 26,I) at which all these cords meet is the *free body* in equilibrium. From the horizontal resolution, with components toward the right positive,

$$10 \cos 10^\circ + Q \sin 25^\circ - P \sin 35^\circ = 0 \tag{1}$$
$$0.4226Q - 0.5736P = -9.848 \tag{2}$$

From the vertical resolution, positive upward,

$$10 \sin 10^\circ + Q \cos 25^\circ + P \cos 35^\circ - 40 = 0 \tag{3}$$
$$0.9063Q + 0.8192P = 40 - 1.736 = 38.264 \tag{4}$$

To eliminate Q, Eq. (2) is multiplied by 0.9063 and Eq. (4) is multiplied by 0.4226.

$$0.9063 \times 0.4226Q - 0.5199P = -8.925 \tag{5}$$
$$0.4266 \times 0.9063Q + 0.3462P = 16.170 \tag{6}$$

When Eq. (5) is subtracted from Eq. (6),

$$0.8661P = 25.095$$
$$P = 28.97 \text{ lb} \tag{7}$$

When this value is substituted in Eq. (2), the result is

$$0.4226Q = 0.5736 \times 28.97 - 9.848 = 16.617 - 9.848 = 6.769$$
$$Q = 16.02 \text{ lb}$$

The unknown force Q of Fig. 26 may be eliminated by resolving along the broken line of Fig. 26,II, which is normal to the direction of Q and, therefore, makes an angle of 25° with the horizontal. On the left, this line makes an angle of 30° with the direction of the force P. On the right, it makes an angle of 35° with the direction of the 10-lb force, and an angle of 65° with the vertical line of the weight.

$$
\begin{aligned}
P \cos 30° - 10 \cos 35° - 40 \sin 25° &= 0 &\quad (8)\\
0.8660P &= 10 \times 0.8192 + 40 \times 0.4226 &\quad (9)\\
0.8660P &= 8.192 + 16.904 = 25.096 &\quad (10)\\
P &= 28.98 \text{ lb}
\end{aligned}
$$

While Eq. (10) is identical with Eq. (7), the labor of calculation is much smaller. After one unknown has been computed by a resolution perpendicular to the direction of the other unknown, any convenient equation may be used for determining the other. For instance, the horizontal and vertical resolutions, which are generally easy to write, may be used for the other unknown and the check. A resolution may be taken perpendicular to the direction of the first unknown. After a resolution has been written *perpendicular* to the direction of one unknown, a resolution *parallel* to that unknown is easy to write, since the angles are complementary, and a sine in one equation is replaced by a cosine in the other and vice versa.

In Fig. 26, resolve parallel to Q.

$$
P \sin 30° + Q + 10 \sin 35° - 40 \cos 25° = 0 \qquad (11)
$$

To write Eq. (11) from Eq. (8), it is only necessary to observe the sign of each term from Fig. 26 and to include the term Q which, if written in Eq. (8), would be $Q \cos 90°$.

$$
\begin{aligned}
Q &= 40 \times 0.9063 - 10 \times 0.5736 - 28.98 \times 0.5 &\quad (12)\\
Q &= 36.252 - 5.736 - 14.490 = 16.02 \text{ lb}
\end{aligned}
$$

After the problem has been solved by Eqs. (8) and (11), the check is most conveniently made by the horizontal or vertical resolution.

Problems

1. A mass of 40 lb on a smooth plane inclined 25° to the horizontal is supported by a rope which makes an angle of 42° with the horizontal. Find the tension in the rope by a resolution parallel to the plane. Find the reaction of the plane by a resolution perpendicular to the plane. Check by a vertical resolution.
   Ans. 17.68 lb; 31.08 lb.
2. A mass of 50 lb on a 20° inclined plane is supported by a rope parallel to the plane which exerts a force of 8 lb and by a rod which pushes horizontally. Find the compression in the rod and the normal reaction, and check.
3. Solve the problem of Fig. 20 by resolutions, and check.
4. Solve Prob. 3 of Art. 21. Check.
5. A mass of m lb is held on a smooth inclined plane, which makes an angle θ with the horizontal, by means of a cord parallel to the plane. Find the tension in the cord and the normal reaction of the plane by resolutions parallel and perpendicular to the plane, and check by a vertical resolution.
   Ans. Tension $= m \sin \theta$; normal $= m \cos \theta$.

6. **A mass of m lb is held on a smooth inclined plane, which makes an angle θ with the horizontal, by means of a horizontal cord. Find the tension in the cord and the normal reaction, and check.**

Ans. Tension $= m \tan \theta$, normal $= m \sec \theta$.

24. Trigonometric Solution. When three concurrent forces are in equilibrium, *the force polygon is a triangle. It is sometimes desirable to draw this triangle and calculate the unknown forces or directions trigonometrically.* When two of the forces are perpendicular to each other, making a right-angled triangle, or when two of the forces are equal, making an isosceles triangle, this method is especially convenient. It is the *best* method when the forces are expressed literally instead of numerically.

Example 1

A 40-lb mass is supported by two cords, one of which is horizontal and exerts a pull of 16 lb. Find the direction of the second cord and the tension which it exerts.

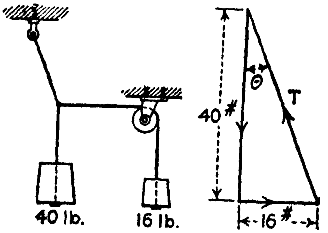

FIG. 27.

The force triangle (Fig 27) is a right-angled triangle of base 16 and altitude 40.

$$\tan \theta = 0.4 \qquad T = 40 \sec \theta$$

Problems

1. A mass of W lb is supported by a rope which makes an angle ϕ with the verical. It is held by a horizontal rope which exerts a pull of P lb. Find the pull P in terms of W and ϕ. Find the tension in the first rope in terms of W and P. Find the weight W in terms of the tension in the first rope and ϕ. Given a frictionless pulley, a known mass, cords, and a carpenter's square, how could this principle be applied to weigh another mass? To secure a known horizontal force?

Ans. $P = W \tan \phi$; $T^2 = W^2 + P^2$; $W = T \cos \phi$.

2. A mass of 48 lb is supported by two cords each of which makes an angle of 20° with the vertical. Find the tension in each cord.
3. Solve Prob. 2 if each cord makes an angle of 15° with the horizontal.
4. Solve Prob. 2 by horizontal and vertical resolutions.
5. Solve Prob. 5 of Art. 23 by means of the force triangle.
6. Solve Prob. 6 of Art. 23 by means of the force triangle.
7. A mass of m lb is held on a smooth inclined plane, which makes an angle α with the horizontal, by a rope which makes an angle β with the horizontal. Find the tension in the rope and the normal reaction by means of the solution of the force triangle.

Ans. Tension $= \dfrac{m \sin \alpha}{\cos (\alpha - \beta)}$; normal $= \dfrac{m \cos \beta}{\cos (\alpha - \beta)}$.

It sometimes happens that the force triangle is similar to a triangle of the space diagram. In that case the magnitudes of the forces are proportional to the corresponding lengths in the space diagram.

Example 2

Figure 28 shows a horizontal rod, 12 ft in length, which is hinged at the left end A. A cord is attached to the right end B and fastened to a point C, which is 8 ft above A. Neglecting the weight of the rod, find the tension in the cord and the compression in the rod when a load of 40 lb is placed at B.

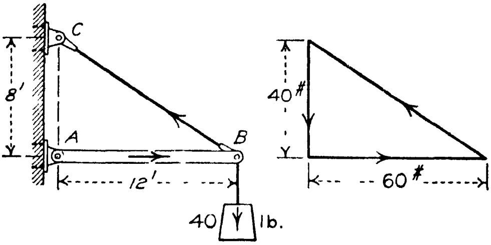

FIG. 28.

The force of 40 lb in the force diagram is homologous to the length of 8 ft in the space diagram. Since the ratio is 5 to 1, the horizontal push in the rod is

$$5 \times 12 = 60 \text{ lb}.$$

In the space diagram the length BC is $\sqrt{208} = 14.422$ ft. The tension in the cord is $5 \times 14.422 = 72.11$ lb.

Problems

8. In Fig. 28, the cord BC is shortened until the point B is 2 ft higher than A. Find the compression in the rod and the tension in the cord. Solve by similar triangles. *Ans.* 60 lb; 66.33 lb.

9. In Fig. 28, the cord BC is lengthened until the point B is 3 ft lower than A. Find the compression in the rod and the tension in the cord. *Ans.* 60 lb; 80 lb.

10. If the length of AB of Fig. 28 is constant and the point C is at a constant distance *directly above* A, show that the compression of the rod is the same for all lengths of the cord BC.

25. Moments. Figure 29 shows a bar OB, which is hinged at O and has a load P applied at B. The load tends to turn the bar about the hinge and does turn it unless balanced by a second force tending to turn in the opposite direction. In Fig. 29,I, the turning effect is greater than in Fig. 29,II. In Fig. 29,III, the force has no tendency to turn the bar around the hinge. In the other positions, the force **P** is said to exert a *moment* on the bar about the hinge. The point about which moment is taken is called the *origin of moments* or the *center of moments*. The origin of moments may be some point in the body, as the hinge O of Fig. 29, or it may be some point entirely outside the body.

Definition. *The moment of a force about a point in its plane is the product of the magnitude of the force multiplied by the perpendicular distance from the point to the line along which the force acts.*

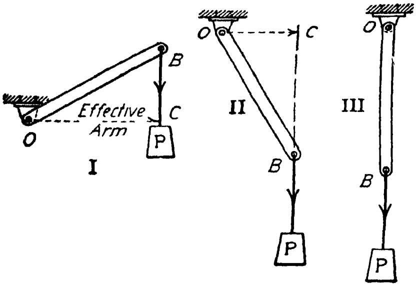

FIG. 29.

In Fig. 29, the moment of the force **P** is the product of the force multiplied by the length OC. If the force is measured in pounds and the distance in feet, the moment is expressed in foot-pounds. To distinguish moment from work, some writers use pound-feet for moment and foot-pounds for work. Though this distinction is desirable, it has not come into general use.

The perpendicular distance from the origin of moments to the line of action of the force is the *effective moment arm.* The length OB, from the origin of moments to the point of application of the force, may be called the *apparent moment arm.* If the length of the apparent moment arm is a (Fig. 30) and the angle between its direction and the direction of the force is α,

$$\text{Effective arm} = a \sin \alpha \qquad (1)$$

$$\text{Moment} = P \times a \sin \alpha$$

Formula III

Counterclockwise moment is generally regarded as positive in mathematical works. In the theory of beams in strength of materials, some authors prefer to take clockwise moment as positive. It is immaterial which sign is used, provided the same convention applies to all the moments of a given equation.

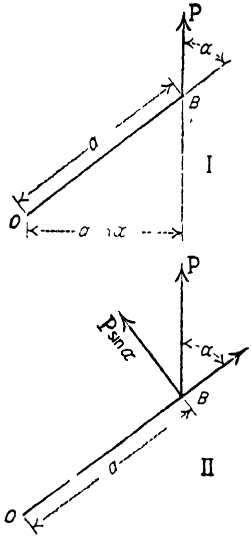

FIG. 30.

In algebraic equations in this book, moment is represented by capital M and mass by lower-case m.

Problems

(*In these problems, construct the space diagram to scale, measure the effective arm, and compare with the computed length.*)

1. A weightless bar AB, 5 ft long, is hinged at the left end A and carries a load of 24 lb on the right end B. Find the moment of the load about the hinge when the right end is elevated 25° above the horizontal.

Ans. M = 108.76 lb-ft, clockwise.

2. Solve Prob. 1 if the right end is depressed 35° below the horizontal.

3. A rectangular board in a vertical plane is 12 in. long and 5 in. wide. The board is hinged at the lower left corner and a force is applied at the upper right corner. Find the effective arm by two methods, and find the moment of the force about the hinge if the 12-in. edges are horizontal and the force is 30 lb at 35° to the left of the vertical upward.

Ans. Effective arm = 13×0.9767 or $12 \times 0.8192 + 5 \times 0.5736$; M = 380.94 lb-in.

4. Solve Prob. 3, if the right end of the board is elevated to bring the 12-in. edges to an angle of 10° with the horizontal. Solve for the effective arm by two methods. Compare with the graphical solution.

5. Solve Prob. 3 if the 12-in. edges are horizontal and the force of 30 lb makes an angle of 20° to the right of the vertical upward. Check.

Ans. M = 286.99 lb-in.

By changing the order of the letters, Formula III may be written

$$M = P \sin \alpha \times a \tag{2}$$

The term $P \sin \alpha$ is the component of the force perpendicular to the apparent moment arm. Equation (2) gives a second definition of moment: *The moment of a force is the product of the length of the apparent moment arm multiplied by the component of the force perpendicular to it (Fig. 30,II).*

The first definition states: *The moment is the product of the entire force multiplied by the component of the arm perpendicular to it.* The second definition states: *The moment is the product of the entire arm multiplied by the component of the force perpendicular to it.*

Since force and apparent arm are vectors, moment is the product of two vectors. It will be shown, in Art. 82, that this product may be represented by a vector normal to the plane of the arm and force.

26. Moment of Resultant. If $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, etc., are forces which make angles α_1, α_2, α_3, etc., with the apparent arm of length a, the sum of the moments of all these forces about the origin is

$$M = P_1 \sin \alpha_1 a + P_2 \sin \alpha_2 a + P_3 \sin \alpha_3 a + \cdots \tag{1}$$

$$M = (P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 \cdots)a \tag{2}$$

The term in the parentheses of Eq. (2) is the sum of the components of all the forces perpendicular to the apparent arm and, therefore, is the component perpendicular to the apparent arm of the resultant of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, etc. Equation (2) gives the moment of the

resultant of these forces and proves the proposition: *The moment of the resultant of several concurrent forces is equal to sum of the moments of the separate forces.*

This proposition is known as *Varignon's theorem,* and it is very useful in practical applications of mechanics. When the moment of a force is calculated by multiplying the apparent arm by the component of the force perpendicular to the apparent arm, the proposition has been used, as the other component then passes through the origin of moments and its moment is zero. The total moment is, then, the moment of the component normal to the apparent arm.

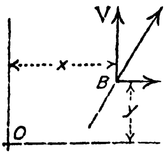

FIG. 31.

It is often advisable to find the moment of a force by means of the sum of the moments of its components. Figure 31 shows a force **P** applied at a point B. The vertical component is **V** and the horizontal component is **H.** If the origin of moments is taken as the origin of coordinates, the moment arm of **V** is x and the moment arm of **H** is y.

$$\text{Moment} = Vx - Hy \qquad \text{Formula IV}$$

Formula IV is especially useful when the coordinates of the origin of moments and of some point on the line of application of the force are given, or when the horizontal and vertical components of the apparent arm may be easily calculated.

This formula is of value in work using formal rectangular coordinates when the coordinates of the origin of moments and of some point on the line of action of the force are known. The student should note that, as is usually the case with mathematical formulas, the situation is presented in the first quadrant and this accounts for the negative sign. In other quadrants some of the terms may be negative, and the moment as computed by the formula is positive if it is counterclockwise. The formula, in spite of the negative sign, says that the moment of the force is the sum of the moments of its components.

Example 1

A force of 40 lb at an angle of 25° to the left of the vertical upward is applied at a point whose coordinates, in feet, are (6,2). Find the moment of this force about the origin of coordinates (Fig. 32).

By horizontal resolution,

$$H = 40 \times 0\ 4226 = 16.904 \text{ lb}$$

By vertical resolution,

$$V = 40 \times 0.9063 = 36.252 \text{ lb}$$

The effective arm of the horizontal component is the vertical length of 2 ft. The effective arm of the vertical component is the horizontal distance of 6 ft Both components turn counterclockwise

$$\begin{aligned} 16.904 \times 2 &= 33.808 \\ 36.252 \times 6 &= \underline{217.512} \\ &\ 251.320 \text{ lb-ft} \end{aligned}$$

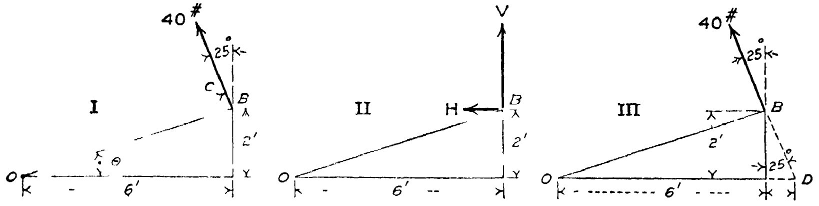

FIG. 32.

The example may be checked by the first definition of moment. The apparent arm, OB (Fig 32,I) is

$$OB = \sqrt{40} = 6.325 \text{ ft}$$

$$\tan\theta = \tfrac{1}{3} \qquad \theta = 18°26'$$

$$OC = OB \cos (25° - \theta) = OB \cos 6°34' = 6.325 \times 0.9934 = 6.283 \text{ ft}$$

$$M = 40 \times 6.283 = 251.32 \text{ lb-ft}$$

Problems

1. A force of 12 lb at 35° to the left of the vertical upward is applied at a point whose coordinates, in feet, are (5,3). Find the moment with respect to the origin of coordinates. *Ans* $M = 49.152 + 20.650 = 69.802$ lb-ft.
2. Check Prob. 1 by means of the first definition of moment.
3. Solve Prob. 1 if the force makes an angle of 35° to the right of the vertical. Check. *Ans.* $M = 28.502$ lb-ft.
4. Find the moment of the force of Prob. 1 about the point whose coordinates are (1,1). Check.
5. Solve Prob. 1 with respect to the points (1,3) and (5,0).
6. Solve Prob. 3 of Art. 25 by means of the horizontal and vertical components of the force.
7. Solve Prob. 4 of Art. 25 by means of the components parallel and perpendicular to the edges of the board.
8. A bar AB, 13 ft long, is hinged at the left end A and supported with the right end B elevated 5 ft above the level of the left end by means of a rope which makes an angle of 30° to the left of the vertical and exerts a pull of 50 lb. Find the moment about A by Formula IV and check by one definition of moment.
9. A bar AB, 10 ft long, has the right end B elevated 20°. It is supported by a rope at B which makes an angle of 10° to the right of the vertical and exerts a pull of 30 lb. Find the moment about a point which is 2 ft above and 3 ft to the

right of A. Also find the moment about a point which is 3 ft below and 5 ft to the right of A. Check both results

Proposition. *In the calculation of moment, any point on the line of action of the force may be regarded as the point of application.*

This proposition is usually regarded as axiomatic. It can be readily demonstrated by the student by showing that in Fig. 33 $P \sin (\alpha - \theta) \times OE$ is equal to $P \times OC$, OC being perpendicular to the line of action of P.

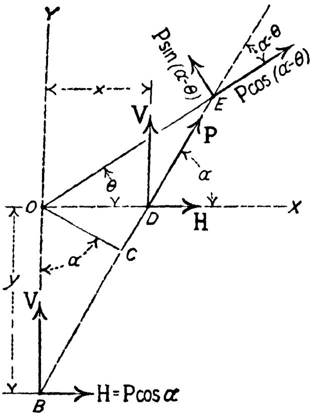

FIG. 33.

The point B of Fig. 33 lies on the Y axis through O. The vertical component of the force has a zero moment arm, and $M = Hy$. At D on the X axis, the horizontal component has a zero moment arm, and $M = Vx$. It is often convenient to extend the line of the force until it intersects a horizontal line through the origin and to calculate the moment by multiplying the vertical component of the force by the distance from the origin to the point of intersection. The line of force may be extended until it intersects the vertical line through the origin, and the moment may be computed by multiplying the horizontal component of the force by the vertical distance from the origin to the point of intersection.

Example 2

In Fig. 32, find the moment about O by multiplying the vertical component of the force by the horizontal distance from O to D

$$OD = 6 + 2 \tan 25° = 6 + 2 \times 0.4663 = 6.9326 \text{ ft}$$
$$M = 6.9326 \times 36.252 = 251.32 \text{ lb-ft}$$

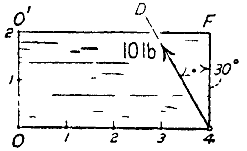

FIG. 34.

Problem

10. In Fig. 34, find the moment of the force about the point O' by multiplying the horizontal arm by the vertical component of the force. Next extend the line of force until it intersects the vertical line through O' and multiply the vertical arm by the horizontal component of the force.

27. Equilibrium by Moments. Since the moment of the resultant of a set of concurrent forces is equal to the sum of the moments of the separate forces, and since the resultant of a set of forces in equilibrium is zero, it follows that *the sum of the moments of a set of con-*

current forces about any origin is zero when the forces are in equilibrium. This affords another method of solving a problem of the equilibrium of concurrent, coplanar forces. The moment equation of equilibrium is

$$M = P_1 \sin \alpha_1 a + P_2 \sin \alpha_2 a + P_3 \sin \alpha_3 a + \cdots = 0 \qquad (1)$$
$$M = (P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \cdots)\, a = 0 \qquad (2)$$

The term in the parentheses in Eq. (2) is the sum of the components, perpendicular to the apparent moment arm of length a, of all the forces which act on the free body. It is evident from this equation, after it has been divided by the apparent arm, that a moment equation for concurrent forces is equivalent to a resolution perpendicular to the apparent arm

Since the moment equation of equilibrium for concurrent, coplanar forces is equivalent to a resolution perpendicular to the apparent moment arm, it would seem that there is no advantage in the use of moment methods. This might be true if all problems were given with the required angles expressed explicitly in degrees. On the contrary, dimensions are frequently so stated that it is much easier to compute the lengths of the effective arms than to find the angles necessary for the resolution method. For a graphical solution by the force polygon, an accurate drawing is required to give the directions of the forces. On this drawing any convenient points may be chosen as origins of moment, the effective arms may be scaled, and the unknown force may be computed by elementary arithmetic. A solution of this kind, which may be called the *graphical-moment method,* is frequently more accurate than the graphical solution by the force polygon. On an actual structure or machine, angles must be measured before resolution methods can be applied. It is generally easier to measure moment arms than to measure angles, and the calculations by moments are shorter and require no tables. It is essential that the engineer should acquire the habit of recognizing moments and moment arms in a structure or machine and should acquire facility in thinking in terms of moments as well as in terms of force components.

Figure 35 represents a weightless bar hinged at O and supporting a load **P** at B. The bar is supported by a cord at B. It is desired to find the tension in this cord. The three forces at B are the load **P,** the tension in the cord **T,** and the compression along the bar. If moments are taken about the hinge O, the line of action of the compression in the bar passes through this point so that its moment is zero. The moment of **T** in a counterclockwise direction must be equal to the moment of **P** in a clockwise direction. The effective moment arm of **T** is the length OD, measured from O perpendicular to the direction of the cord. The moment arm of **P** is the perpendicular distance from the hinge to its line of action, the length OC of Fig. 35. To find the perpendicular distance from a point to a straight line, it is only necessary to measure the shortest distance. This may be done with a tapeline or rule. In the actual machine or structure, it is much easier to measure these lengths than it is to determine the angles necessary for resolution equations.

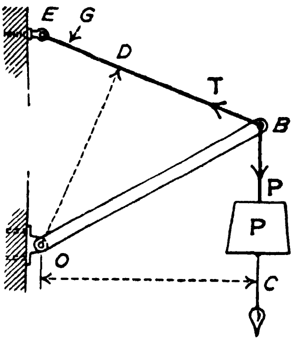

FIG. 35.

The moment equation for finding the tension T is

$$T \times OD = P \times OC \qquad (3)$$

To find the compression in the bar OB of Fig. 35, the point E may be used as the origin of moments, or any other point, as G, on the line EB.

When the origin of moments is taken on the line of action of a force, the moment arm of that force is zero. In the solution of a problem of equilibrium, it is desirable to take at least one origin of moments on the line of one of the unknown forces to avoid simultaneous equations.

Problems

1. A beam, as in Fig. 35, is 10 ft long, and the fixed point E is 8 ft directly above O. The right end of the beam is 2 ft higher than O, and the load **P** is 200 lb Find the tension T and the compression in OB by the graphical moment method. Use a scale 1 in. = 2 ft and take origins at O and E. Check both results by moments about a point 2 ft above O.
2. Solve Prob. 1 when the end B is 2 ft lower than O
3. In Prob. 1, solve trigonometrically for the angles. Calculate OD and solve for **T** by moments to four significant figures. Calculate the perpendicular distance from E to OB and solve for the compression in the beam. Check by a vertical resolution.
4. In Fig. 28, find the compression in AB by moments about C; calculate the moment arm and find the tension in the cord.

Example

A mass of 40 lb is supported by two cords, one of which makes an angle of 30° to the right of the vertical and exerts an unknown force **Q**. The second cord is attached to a fixed point A, which is 12 ft higher and 5 ft to the left of the point of attachment of the two cords to the mass. Find the tension **Q** by moments about A.

Find the tension in the other cord by horizontal resolutions. Check by vertical resolution.

(*The student will make a sketch.*)

$$Q \sin 30° \times 12 + Q \cos 30° \times 5 = 40 \times 5$$
$$10.33Q = 200$$
$$Q = 19.361 \text{ lb}$$
$$0.5Q = \tfrac{5}{13}T$$
$$T = 1.3 \times 19.361 = 25.169 \text{ lb}$$

The student will make all the calculations, including all omitted steps, and will complete the check.)

28. Independent Equations. As was stated in Art. 21, the proper number of unknowns for a concurrent, coplanar problem of statics is two, and in each problem of this class in the preceding articles there have been just two unknowns. These have been

1. An unknown magnitude and an unknown direction of one force.
2. An unknown magnitude of one force and an unknown direction of another.
3. Two unknown magnitudes.
4. Two unknown directions.

With two unknowns, two independent equations are required to solve each problem. Figure 36 shows a force AB. Its orthographic component along a direction OC is A_1B_1, and its orthographic component along a direction OD is A_2B_2. If a third orthographic component were drawn, its length could be expressed in terms of A_1B_1, A_2B_2, and known angles. Consequently, if three resolution equations were written for a problem of concurrent, coplanar forces, any one of these could be derived algebraically from the other two. Only two resolution equations, therefore, are independent. Expressed in another way, the solution of a problem of equilibrium may be regarded as equivalent to the determination of the magnitude, direction, and position of a single resultant or equilibrant, and the number of unknowns in a solvable problem falling under any classification will be the number of elements necessary to describe completely a single force in that system. When the plane of the force is given and its position in this plane is known to be at the point of application of all the forces, only the magnitude of this resultant and its direction in the known plane remain to be determined. There can be only two unknowns, unless there are other conditions of the problem which are not based on mechanics.

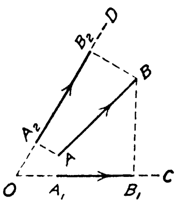

FIG. 36.

Example

A mass of 60 lb is supported on a 30° plane by a force **F** down the plane and a horizontal force **H** which tends to pull the body up the plane. Find these forces and the normal reaction of the plane.

There are three unknowns and the problem is indeterminate unless another condition is known. Suppose it is known that **F** is equal to 40 per cent of the normal force. This reduces the unknowns so that the problem can be solved.

A vertical resolution eliminates the unknown **H**.

$$\begin{aligned} 0.866N - 0.5 \times 0.4N - 60 &= 0 \\ 0.666N &= 60 \\ N &= 90.09 \text{ lb} \end{aligned}$$

A horizontal resolution gives

$$\begin{aligned} H - 0.5N - 0.866 \times 0.4N &= 0 \\ H = (0.5 + 0.3464)N &= 0.8464N \\ H = 0.8464 \times 90.09 &= 76.252 \text{ lb} \end{aligned}$$

Check by a resolution parallel to the normal.

The two possible independent equations may be written in any one of three ways:

1. Two resolution equations.
2. Two moment equations.
3. One resolution equation and one moment equation.

A moment equation is equivalent to a resolution perpendicular to the line joining the origin of moments to the point of application of the forces. If two moment equations are written and the two origins of moment and the point of application of the forces lie on the same straight line, the equations are not independent After each moment equation has been divided by the length of its apparent moment arm, the two are identical, and the resulting equation is equivalent to a resolution perpendicular to the moment arm. *When two moment equations are written for a problem of concurrent, coplanar forces, the two origins of moment and the point of application of the forces must not lie in the same straight line. When one resolution and one moment equation are written, the resolution must not be perpendicular to the line joining the origin of moments with the point of application of the forces.*

A resolution perpendicular to the direction of an unknown force eliminates that force. A moment about a point in the line of action of an unknown force eliminates that force.

29. Questions

1. Give three methods for finding the resultant of two concurrent forces.
2. Give two practical methods for finding the resultant of three or more concurrent, coplanar forces.

3. Give two definitions of the moment of a force and show how one definition is derived from the other.
4. Name four ways of solving a problem of the equilibrium of concurrent, coplanar forces.
5. In the solution of a problem of equilibrium by resolutions, how may the resolution be taken to eliminate one unknown?
6. In the solution of a problem of equilibrium by moments, how may an unknown be eliminated?
7. When a problem of equilibrium is solved by one resolution and one moment equation, what precaution is necessary? What precaution is necessary when two moments are used?
8. When is it convenient to calculate moment by resolving both the force and the apparent arm into components?
9. What is meant by the *graphical-moment method* for the solution of a problem of equilibrium?

30. Miscellaneous Problems

1. Find the magnitude and direction of the resultant of 60 lb at an angle of 20° with the horizontal, 40 lb at 60° with the horizontal, and 10 lb 40° (counterclockwise) from the vertical. Check graphically.
2. To the system of three forces given in Prob. 1 add the *equilibrant, i.e.*, the equal and opposite of the resultant, and check by resolving on some axis not used in finding the resultant.
3. Find the resultant of 60 lb at 20° with the X axis and 80 lb at 110°. Solve by horizontal and vertical resolutions. Solve again by a more convenient pair of axes. Check by a force triangle.
4. A 50-lb mass is attached at a point 5 ft from one end of a cord 11 ft long, and the ends of the cord are attached to a horizontal bar at such a distance apart that the point of attachment of the mass is 3 ft below the bar. Find the tension in each of the two parts of the cord. Solve by resolutions and check graphically. *Ans.* 46.53 lb; 43.65 lb.
5. A cord 10 ft long is fastened to two points at the same level and 8 ft apart. From a small smooth ring on the cord a 60-lb mass is suspended. Find the tension in the cord.
6. A force of 100 lb makes an angle of 30° with the horizontal. Find its rectangular component on an axis making an angle of 50° with the horizontal. Find its horizontal and vertical components, and find the sum of the components of the horizontal and vertical components on the 50° axis. (This sum should check with the component of the 100-lb force on the same axis.)
7. A mass of W lb is hung on a rope which makes an angle α with the vertical and is held in position by a second rope. What should be the direction of the second rope if it is to have the minimum possible tension? Solve graphically and check by considering the moments of all forces with respect to a point on the first rope. What should be the direction of the second rope if the tension in the first rope is to be a minimum?
8. An unknown mass is placed on a smooth inclined plane which makes an unknown angle with the horizontal. It is found that it can be held in place either by a horizontal force of 30 lb or by a force of 24 lb parallel to the plane. Find the four unknown quantities. (Two separate free-body diagrams should be drawn. From each of the diagrams, two independent equations can be

written and the four unknowns can be determined. The situation will be confused by an attempt to put all the data into one space diagram.)

9. Solve Prob. 8 graphically. (The situation may be clarified by superimposing the force diagrams for the separate conditions of the problem.)

10. The resultant of a force of 400 lb and a force of 1,300 lb is a force of 1,500 lb. What is the angle between the forces? *Ans.* 67°23′.

11. A mass of 100 lb is supported by a cord making an angle of 30° to the left of the vertical, a second cord at 40° to the right of the vertical, and a third cord horizontally to the left. If the horizontal cord exerts a force equal to 0.5 times the smaller of the other two, what is the tension in each of the three cords? (Study a force polygon until the condition becomes clear and then solve by vertical and horizontal resolutions.) *Ans.* The horizontal force is 24.3 lb.

12. Find the force **P** to move the block in Fig. 1 if α is 20° and **F** cannot exceed $0.65N$. Find also the smallest force **P** to move the block and the corresponding value of α with the same condition for **F**.

13. A small 15-lb mass at the upper end of a light rod 17 in. long is placed against a smooth vertical wall with the rod in a vertical plane and its lower end 8 in. from the wall. Find the coefficient of friction between the end of the rod and the floor necessary to prevent slipping. Would the rod stand with a smaller coefficient at the floor if the wall were rough?

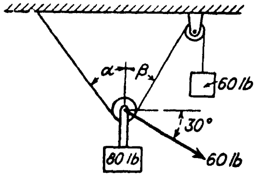

FIG. 37.

14. Figure 37 shows an 80-lb **mass hung** from a smooth pulley free to **run along** a cord where the tension is 60 lb. The pulley is pulled downward at 30° below the horizontal by a force of 60 lb. Write two resolution equations and solve for the two unknown angles. *Ans.* The situation is impossible. An attempt to do a graphical solution will show the nature of the difficulty.

CHAPTER 3

APPLICATION OF FORCES

31. Forces. For the system of forces discussed in Chap. 2, it appeared that for any given problem a direct procedure can be set up leading to a definite solution and that it can be determined by rule whether or not any given problem has a definite solution. It will presently be seen that the same things may be said of the more complex systems, but it must not be concluded that mechanics is a game wherein it is necessary only to know the rules. Applied mechanics, the serious business of the designer of machines and structures, requires, above all, a clear conception of how forces arise and how they are applied to the bodies under consideration. Every practical problem of applied mechanics involves the study of a set of forces and their effects on matter. No mathematical method, however powerful, is likely to be fruitful unless it is used with a clear understanding of what is known about the problem and of what can be assumed without destroying the usefulness and correctness of the conclusions.

Forces are classified as *mechanical forces*, or forces which are exerted on a body by other bodies in actual contact with it, and *field forces*, or forces which somehow act at a distance as do gravitation, magnetic and electrostatic forces, and the inter- and intraatomic forces.[1] For the sake of simplicity, no forces are considered in this text except contact forces and gravitation. Gravity forces appear only as the weights of the bodies under study. Fictitious forces such as "inertia forces," "Coriolis force," etc., are discussed in Chap. 19. There are a few problems dealing with simplified magnetic fields.

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[1] After centuries of study, the nature of field forces is still obscure. It is fashionable at present to consider gravitation not as a force but as something inherent in space, time, and matter which causes masses to accelerate toward each other (an opinion widely held before the time of Newton) and to consider other field forces as actual forces. The distinction seems to arise from the circumstance that gravitation cannot be "cut off" by intervening matter. It is now generally agreed that bodies are never in actual contact but are merely in such close proximity that the intermolecular and interatomic forces become significant, so that "contact forces" are really field forces.

32. Tension, Compression, and Shear. When forces are applied to a body, *stresses* are produced in it. The stress which resists stretching or elongation is called *tensile stress.* A vertical rope (Fig. 38) which supports a 10-pound mass is subjected to a downward force of 10 pounds at one end and an upward force of 10 pounds (plus the weight of the rope) at the other end. The rope is said to be in *tension*, and the *total tensile stress* at any section is 10 pounds plus the weight of the rope below the section. The *unit tensile stress* (which will be considered in the course in Strength of Materials) is the total tensile stress divided by the area of the cross section. A body is in tension when it is subjected to a pair of equal forces which are opposite in direction, along the same line, and *away* from each other.

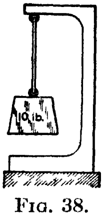

FIG. 38.

Figure 39 shows a 50-pound mass resting on a short block. The block is subjected to a downward push of 50 pounds at the top and an upward push of 50 pounds (plus its own weight) from the support at the bottom. The block is in *compression* and is subjected to *compressive stress.* A body is in compression when it is subjected to a pair of equal forces which are along the same line, opposite in direction, and *toward* each other. A body in compression is shortened along the line of the forces.

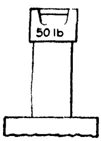

FIG. 39.

In Fig. 40, the block *AB* is said to be in *shear* and subjected to *shearing stress.* A body is in shear when it is subjected to a pair of equal forces which are opposite in direction, and which act *along parallel planes.*

In Fig. 38, the rope is in tension, the horizontal portion of the frame at the top is in shear (combined with bending), and the vertical post is in compression (combined with bending).

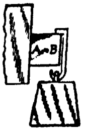

FIG. 40.

33. A Rigid Body. A solid body, which suffers little change of form when subjected to considerable force, is called a *rigid body.* All bodies are elastic, so that there is some change in form or dimensions when force is applied; but these changes are frequently so small as to be negligible, except for measurements requiring the greatest accuracy. Such bodies are regarded as rigid. Figure 41,I shows a beam supported at the middle with a load on each end. The beam is somewhat bent, but the amount of bending may be so small that the distance of the loads from the vertical line through the support is not materially changed. Figure 41,II shows a lighter beam, in which the amount of bending is sufficiently great to change materially its form and dimensions. If, however, the beam is considered as it is now loaded, it may be regarded as a rigid body, but one of different form and dimensions from what it would have with a different loading.

In elementary calculations of *mechanics*, no allowance is made for slight elastic deformations. In that branch of *applied mechanics* which is called *mechanics of materials* or *strength of materials*, these deformations are taken into account.

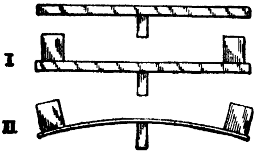

FIG. 41.

34. Friction. When two solid bodies are in contact with a definite pressure between them, each is capable of exerting a limited force on the other in any direction tangent to the surface of contact. This force is called the *force of friction*, or simply *friction*, and its maximum value depends more on the nature and condition of the surfaces in contact and on the total pressure between them than on the velocity of sliding or the unit intensity of the pressure. Figure 42 shows a small body in equilibrium on a horizontal plane with a force P acting parallel to the plane. The force diagram for the system is shown at the right in this figure. If P is below the limiting value, the friction force F remains equal to P and the body remains at rest. If P is now increased, F increases with it until the limiting value is reached and the body starts to move. After motion starts, F drops slightly below the value it had when the body started and then remains practically constant at this smaller value, regardless of the value of P. If P is reduced until it is equal to the new value of F, the body continues to move with uniform velocity; if P is made larger or smaller than the value of F, the body speeds up or slows down accordingly. In either of these latter two cases, the force diagram $PWFN$ at the right of the figure would not close and the force system would have a resultant causing the acceleration.

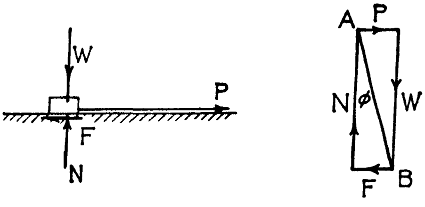

FIG. 42.

35. Friction of Dry Surfaces. Early experiments on the friction of dry surfaces were made by Morin and Coulomb, and the following statements about the friction between such surfaces are called *Coulomb's laws*.

1. The friction is directly proportional to the (total) normal pressure between the surfaces. (*Friction* here means the limiting value of the force.)

2. The friction is independent of the velocity of rubbing.

3. The friction between two surfaces at rest (relatively) is greater than when they are in motion.

4. The friction is independent of the area of the surfaces in contact for any given normal pressure, *i.e.*, it is independent of the unit intensity of the normal pressure.

5. The friction depends upon the nature of the surfaces in contact.

The experiments from which these laws were deduced covered only a small range of low pressures and velocities, and the "laws" have been found by later experimenters to be considerably in error. Some of these disagreements are discussed below. Coulomb's laws are still useful in mechanics, however, as they permit a simple algebraic statement of the relation between the limiting friction force and the normal pressure with which it is associated. The ratio of the limiting friction force to the normal pressure at the surface is called the *coefficient of friction*, represented in this text by the Greek letter μ (mu).

$$F = \mu N \qquad \text{or} \qquad \mu = \frac{F}{N} \qquad \text{Formula V}$$

Here F is the limiting friction force and N is the normal pressure. In practical work, coefficients of friction must be chosen with caution and then amended by a factor of safety. Handbooks give tables of coefficients in terms of limiting values, and the designer's choice must be made on the basis of whether he wishes sliding to take place or not.

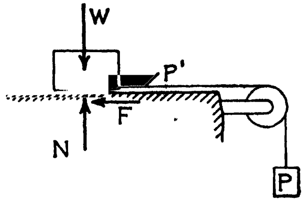

FIG. 43.

Example

In Fig. 43, the mass W weighs 12 lb and the force parallel to the surface required to start the body is 5.4 lb. The force required to keep the body in motion after it is started is 4.2 lb. Find the coefficient of *starting friction* and the coefficient of *moving friction.*

Since the plane is horizontal, the normal force is equal to W.

$$\mu = \frac{5.4}{12} = 0.45$$

$$\mu = \frac{4.2}{12} = 0.35$$

Problems

1. A block of wood weighing 60 lb resting on a horizontal floor required a horizontal force of 23.4 lb to start it and thereafter a horizontal force of 19.5 lb to keep it in motion at 4 ft per sec. Find the coefficients of friction, static and kinetic.
Ans. Static coefficient = 0.39; kinetic coefficient = 0.325.
2. What pull parallel to the plane will be required to drag the body of Prob. 1 up a 30° inclined plane of the same material as the floor? What force is required parallel to the plane for uniform motion down the plane?
Ans. 46.89 lb and 13.11 lb up the plane.
3. A mass of 60 lb is pulled up a 30° plane by a force of 42 lb parallel to the plane. Find the coefficient of friction. *Ans.* $\mu = 0.231$.
4. A mass of 60 lb is pulled along a horizontal plane by a force of 24 lb at an angle of 20° with the horizontal. Find the coefficient of friction.
Ans. $\mu = \frac{22.553}{51.792} = 0.435$.
5. What force at an angle of 20° with the horizontal is required to drag a 40-lb body along a horizontal plane where the coefficient of friction is 0.28?
Resolving horizontally,
$$P \cos 20° = F = 0.28N$$
Resolving vertically,
$$N = 40 - P \sin 20°$$
Ans. $P = 10.81$ lb.
6. A mass of 50 lb is just pulled up a 30° plane at a constant speed by a force of 40 lb at an angle of 45° with the horizontal Find the coefficient of friction.
Ans. $\mu = 0.417$.
7. What pull up the plane will just permit the 60-lb mass of Prob. 3 to slide down the plane at a uniform speed after it has been started? *Ans.* 18 lb.
8. What pull at 45° with the horizontal upward will just permit the 50-lb mass of Prob. 6 to slide down the plane at a uniform speed after it has been started?
Ans. $P = 8.11$ lb.

36. Variation of the Coefficient of Friction. If experiments are made with an apparatus like that shown in Fig. 42, keeping the area in contact with the plane constant and making all trials at a constant speed of sliding, it will be found that as W is increased the coefficient of friction at first decreases gradually and then rises rapidly until seizing occurs. On examination, a pair of surfaces which have seized are usually found to be galled and stuck together. There is little doubt that the rising coefficient of friction indicates incipient galling. If the coefficient under light pressure and the galling pressure have been determined (at a given speed of sliding), the friction force can be estimated fairly accurately at any pressure by the equation $F = \mu N^{0.97}$, where μ is the coefficient at low pressures and N is the total pressure between the surfaces. This equation should not be used above three-fourths of the galling pressure. For most purposes the first of Coulomb's laws is sufficiently accurate without a pressure correction.

The second law states that the friction is independent of the velocity of rubbing. This law is not sufficiently accurate for anything other than rough estimates. Westinghouse and Galton (in England)[1] got the following average values for steel wheels on steel rails:

Speed, in miles per hour	10	15	25	38	45	50
Coefficient of friction	0.110	0.087	0.080	0.051	0.047	0.040

Kilgore and Veley[2] found that the coefficient of friction for automobile tires on wet brick roads dropped from 0.561 at 4 miles per hour to 0.299 at 10 miles per hour and decreased more gradually for higher speeds. On dry penetration macadam, the coefficient dropped from 0.657 at 4 miles per hour to 0.491 at 9 miles per hour. The velocities above were all velocities of sliding or skidding. The resistance of a wheel to skidding (initial) also decreases as the *rolling velocity* increases. This will be discussed later.

Ordinary tests with an apparatus like that shown in **Fig. 43** indicate a sudden drop in the friction coefficient from the static condition to the moving condition, but more careful experiments with controlled speeds have shown that there is a gradual, continuous decrease in the coefficient at speeds from 0.0002 to 0.001 feet per second.[3]

37. Angle of Friction. Figure 42 shows the simple case of a body on a rough horizontal plane being pulled by a force parallel to the plane. In the vector diagram at the right, the vector AB is the resultant of **P** and **W** and the vector BA is the resultant of the friction force and the normal force. (No arrows are shown on these vectors and the directions are indicated by the order of reading the letters at the ends of the line. This is a common and very convenient convention when a vector represents forces which are equal, opposite, and along the same line of action.) If the block is on the point of moving along the plane, the angle ϕ in the force diagram is the *angle of starting friction*. If the block is moving uniformly along the plane, ϕ is the *angle of moving or kinetic friction.* From the figure it is clear that $\tan \phi = F/N$, or $\sin \phi = F/R$, where R is the resultant of the friction and the normal forces. *When the friction force has its limiting value, the angle between the normal force and the resultant of the friction and normal forces is called the angle of friction.* This angle is the key to graphical or trigonometric solutions of problems involving friction.

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[1] *Proc. Inst. Mech. Engr.*, 1879.

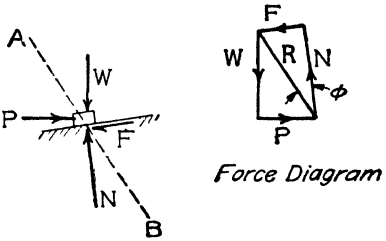
[2] *Ohio State Univ. Studies*, 1939.

[3] Prof. Fleming Jenkin, *Proc. Roy. Soc.*, 1877.

In analytical solutions it is often convenient to eliminate the normal force and the friction force by resolving in a direction perpendicular to their resultant, *i.e.*, by using an axis at the angle ϕ with the friction force.

Example 1

Find the horizontal force P required to start a body of weight W up an inclined plane. The space diagram is shown at the left in Fig. 44. If we know the inclination of the plane and the coefficient of friction, F can be replaced by μN and the two unknowns are the magnitudes of P and N. In the space diagram, draw the dotted line AB making the angle of friction with the direction of **N**. In the force diagram, lay off **W** to scale. From the origin of **W**, draw the line **R** parallel to AB, and from the terminus of **W** draw **P** to the right until it intersects **R**. Scale **P** for a numerical answer. To complete the force diagram, draw in **N** perpendicular to the plane and **F** parallel to the plane, as shown. **R** is a vector in this diagram and makes the angle ϕ with **N**. Reading upward to the right, it is the resultant of **N** and **F**. Reading downward to the left, it is the equal and opposite resultant of **W** and **P**.



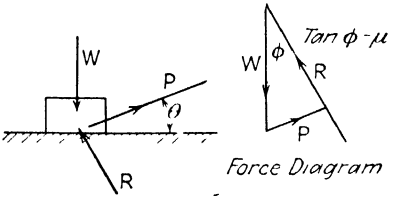
Force Diagram

FIG. 44.

Example 2

Find the minimum force required to drag a body along a horizontal plane and the angle at which it must act.

In Fig. 45, the weight W is laid off as a vertical line of known length. The resultant of the normal and the friction makes an angle ϕ with the vertical. The tangent of ϕ is the coefficient of friction. Through one end of the vector **W** of the force diagram, a line of indefinite length is drawn at an angle ϕ with the vertical. This line gives the direction of the resultant of the friction and the normal. A line from the other end of **W** to the line of this resultant represents the force **P**. The length of this line is the least when it is perpendicular to the line of the resultant. Since the resultant makes an angle ϕ with the vertical, the line **P** perpendicular to the resultant makes the same angle with the horizontal. The pull is a minimum when its angle with the horizontal plane is the angle of friction.



Force Diagram

FIG. 45.

38. Fluid Friction. When layers or laminae of a fluid (liquid or gas) are made to glide along each other or when a fluid moves along a solid surface, a resistance to the motion is set up, which is termed *fluid friction.* There are two clearly distinguishable cases, (*a*) where the layers (probably one molecule thick) actually do glide along each other without mixing and (*b*) where parts of each layer are continually moving out of line and joining with the adjacent layers which are moving with greater or lesser velocities. Case *a* is called *laminar flow* or *viscous flow* (even though the fluid may have very low viscosity) and case *b* is called *turbulent flow*. Of course there are cases of mixed type intermediate between the two, either with some parts of the surface having one kind of flow and other parts the other, or areas on the surface where the type of flow varies rapidly with time. In nearly every case where a fluid moves along a smooth solid there is a laminar layer adjacent to the solid.

Suppose two smooth parallel solid surfaces are separated by a layer of liquid and one of the solid surfaces is moved relative to the other with a constant velocity and with the liquid layer remaining uniformly thick. The force required to maintain the constant motion is directly proportional to the area of the moving surface, directly proportional to the viscosity of the liquid, and inversely proportional to the thickness of the liquid layer if the layer is thin enough so that the flow is laminar. This is the law of viscous friction substantially as stated by Newton, and the law serves to define the numerical factor known as the *viscosity* of the fluid. In algebraic form the law is $F = \mu VA/t$, where F is the force to maintain the constant velocity V, A is the area of the surfaces, t is the thickness of the liquid layer, and μ is the *coefficient of viscosity*. (It should be noted that this μ is not a simple numerical ratio as was the same symbol used for the coefficient of friction but is a quantity having dimensions, with its numerical value dependent on the units of force, time, and length used. Since the coefficient of viscosity is not used in any problems in this text, no confusion is likely to arise from using μ in the equation above. It is the customary symbol for absolute viscosity in works on fluid mechanics.) It has been found that gases and most liquids follow this law with a high degree of exactness. Some colloid solutions and liquids with large complex molecules which do not follow the law closely are called *non-Newtonian liquids*.

Surfaces in relative motion completely separated by a liquid film between them are called *lubricated surfaces*. The thicknesses of film which can be maintained under various conditions of temperature, velocity, pressure, etc., are covered in works on the mechanics of fluids and on the theory of lubrication and are beyond the scope of this text. The discussion above, however, along with the general rule that the viscosity of a liquid decreases with temperature, may enable the student to understand the so-called *laws of lubricated surface friction* and why they seem to be almost exactly the opposite of the laws of dry surface friction. Paralleling the order used before, these laws for lubricated surfaces are:

1. The friction resistance is independent of the total normal pressure.[1]

2. The friction increases with the velocity of sliding.

3. The friction between two surfaces relatively at rest is zero (provided the film is maintained).[2]

4. The friction is directly proportional to the area of the surfaces.

5. The friction is independent of the nature of the surfaces.

Between the conditions of dry surfaces and lubricated surfaces lies the condition of partially lubricated surfaces on which there is oil or grease insufficient to prevent actual contact of the solids. Surfaces of this kind behave much like dry surfaces but with greatly reduced coefficients of friction. In this text, as in most discussions of the subject, the force problems of lubricated surfaces are treated exactly as though the surfaces were dry, but of course the coefficients of friction may be considerably smaller. The coefficient of friction for a lubricated surface may easily be smaller than 0.001 under good conditions and compares favorably to that of a good ball or roller bearing. Ball and roller bearings have their principal advantages in their ability to maintain closer alignments and to run at higher speeds rather than in their reduction of friction, except on starting, in which condition they show a great advantage.

39. The Smooth Surface. No surface is absolutely without friction when another surface is moved along it, but in many problems of mechanics the concept of a smooth surface is used because the friction force is either negligible or undependable. When surfaces in contact are subjected to rapid vibrations or jarring they may behave as though the coefficient of friction were zero or even negative. The force exerted by or on a smooth surface must be normal to the surface if the surface is plane or normal to the tangent at the point of contact if the surface is curved.

Statics deals chiefly with conditions of equilibrium. The position of equilibrium for perfectly smooth bodies is approximately midway between the limiting positions if the bodies were rough. For this reason the elementary discussions of statics assume frictionless bodies.

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[1] The normal pressure may change the film thickness, and this would cause a corresponding change in the force. Also, the viscosity of liquids is slightly increased by pressure so that the friction actually increases with pressure.

[2] In a bearing the film is never maintained when the bearing stops unless the lubricant is being forced in under pressure, so there is usually a relatively high starting resistance.

To demonstrate experimentally the mechanics of ideal frictionless surfaces, apparatus may be provided with wheels or rollers, as shown in Fig. 46.

40. The Smooth Hinge. Two bodies are frequently connected as in Fig. 47. A cylindrical pin passes through cylindrical holes in each of the bodies B and C. The pin may be fixed in one of the bodies but must be free to turn in the other. This form of connection is called a *hinge*. One body can rotate relative to the other in a plane which is perpendicular to the axis of the pin. In a smooth hinge or pin connection the force between the pin and the hollow cylinder is normal to the curved surfaces at the line of contact. The force is, therefore, along a line through the axes of the pin and the hollow cylinder, and in a plane normal to these axes.

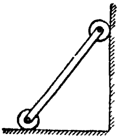

FIG. 46.

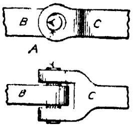

FIG. 47.

In Fig. 47, the hollow cylinder in body C is drawn much larger than the pin A. In practice, the pin is made to fit the cylinder with little clearance.

A bar with a smooth hinge in each end, as in Fig. 48, is called a *link*. A link transmits force in the direction of the line joining the two hinges. A single link may be in tension, as in Fig. 48, or in compression. A series of links with hinges parallel is equivalent to a flexible cord and transmits tension alone.

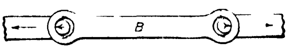

FIG. 48

41. The Smooth Pulley. A circular disk mounted on a smooth pin at its center and having a groove around the periphery is a *smooth pulley*. In statics, a cord passing over such a pulley has the same tension on both sides. In dynamics the tensions on the two sides may be different because of the friction between the cord and the pulley. Sometimes this difference is neglected, and the pulley is called a smooth, light pulley. A smooth roller is like a smooth pulley except that its periphery rolls along a surface instead of along a rope or cord. The force exerted by a smooth roller or on a smooth roller is radial (neglecting rolling resistance to be discussed later). Figure 46 shows a bar provided with a smooth roller at either end, with one roller on a horizontal floor and the other on a vertical wall. The situation would be exactly the same if the floor and wall were smooth and the rollers were omitted. The bar as shown cannot possibly be in equilibrium, but it could be put in equilibrium by a single force of suitable magnitude and direction applied anywhere along the bar.

42. The Flexible Cord. Figure 49,I shows a *stiff* rope supported at the middle This rope has some rigidity It is similar to a very flexible beam Figure 49,II shows a *perfectly flexible* cord The cord takes the form of the support at the top and hangs vertically downward at the ends An ideal flexible cord offers no resistance to bending A flexible cord or rope can exert force only in the form of tension in the direction of its length When a flexible cord forms a part of a structure or piece of apparatus the position and direction of the force which it exerts are definitely know

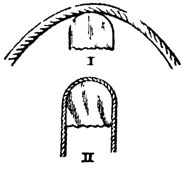

FIG 49

In Fig 50, two flexible cords are attached to the beam AB. The direction of the tension exerted by each cord on the beam is shown by the arrow; the position of the force in each is known to be along its axis On the other hand, the direction of the force exerted by the beam at B cannot be determined by inspection of the diagram but must be calculated from the direction and magnitude of the forces in the cords

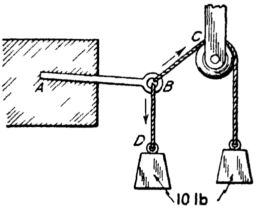

FIG 50

43. The Free Body. In every problem of mechanics with any practical significance the forces which are considered act on some definite body or portion of a body and, as was stated before, this body or portion of a body is called the *free body.* In the solution of the problem it is necessary to designate clearly the free body and to account for all the forces which act on it by the attraction of gravity and by the other bodies in contact with it.

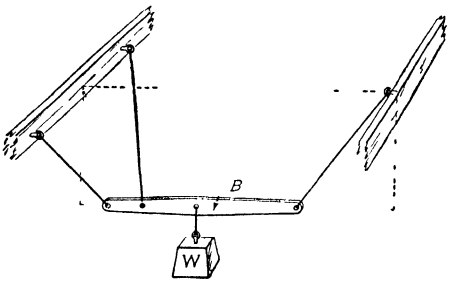

FIG 51

In Figure 51, for example, the mass W may be taken as the free body. The forces which act on it are the attraction of the earth downward and the upward pull of the vertical cord Again, the beam B may be taken as the free body. The forces which act on B are its weight, which is the attraction of the earth at its center of gravity, the downward pull of the vertical cord, which is equal to W, and the tensions in the three inclined cords which support it. The beam B, the mass W, and the cord which connects them may be considered together as the free body. The external forces are the weight of B at its center of gravity, the weight of W at its center of gravity, the weight of the short vertical cord (if it is not negligible), and the tensions in the three inclined cords. The tension in the vertical cord is an *internal* force which is not now considered. A portion of the beam to the right of the section at B may be taken as the free body. The forces are now the weight of the portion at its center of gravity, the tension in the rope attached to the right end, and the shear, tension (or compression), and moment in the beam itself at the section. (This is an important statics problem preceding the problem of finding the stresses at the section, which is a problem of strength of materials.)

CHAPTER 4

NONCONCURRENT, COPLANAR FORCES

44. Resultant of Nonparallel Forces. Figure 52 shows three coplanar forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, which are applied to a rigid body at A_1, A_2, and A_3, respectively. It is assumed as an axiom: *A force may be regarded as applied at any point along its line.* Since the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ intersect at B, they may be treated as concurrent at that point.

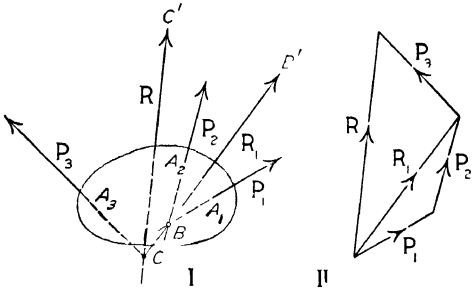

FIG 52

The direction and magnitude of the resultant of these two forces are given by the line $\mathbf{R}_1$ of the force triangle of Fig. 52,II. This resultant is applied along the line BB', which passes through the intersection B of the space diagram and is parallel to the force $\mathbf{R}_1$ of the force triangle. The third force $\mathbf{P}_3$ and the resultant of the first two intersect at C, which is outside the rigid body. The resultant of $\mathbf{R}_1$ and $\mathbf{P}_3$ at C is $\mathbf{R}$ of the force polygon. This is the resultant of all three forces and acts along the line CC''.

The force diagram shows that the *direction* and *magnitude* of the resultant of a set of nonconcurrent, coplanar forces may be found from the force polygon in exactly the same way as if the forces were concurrent.

When forces are concurrent, the location of the resultant is known, since it must pass through the point of application of all the forces. When forces are nonconcurrent, the location of the resultant must be determined. The combination of the force polygon with the process of successive intersections on the space diagram as shown in Fig. 52 is the graphical solution of the problem. Finding these intersections by trigonometric computations would be possible, but obviously laborious and tedious. This labor can always be avoided by making use of the following proposition: *The moment of the resultant of a set of nonconcurrent, coplanar forces about any point in the plane is equal to the sum of the moments of the separate forces about that point.*

This proposition follows from Varignon's theorem. In Fig. 52 the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are concurrent at *B* and their resultant is $\mathbf{R}_1$. Therefore, from the theorem the moment of $\mathbf{R}_1$ about any point in the plane is equal to the sum of the moments of $\mathbf{P}_1$ and $\mathbf{P}_2$, and the argument can be extended to any number of forces.

Problems

1. A force of 20 lb at an angle of 60° to the right of the vertical is applied at the point (5,0); a force of 15 lb at 15° to the right of the vertical is applied at the point (3,2); and a force of 12 lb at 25° to the left of the vertical is applied at the point (−2,3). Find the magnitude and direction of the resultant. Find the distance of the resultant from (0,0), and also find where it crosses the *X* axis. The graphical solution may be drawn to the scales of 1 in. = 2 units on the space diagram, and 1 in. = 10 lb on the force diagram (Fig. 53).

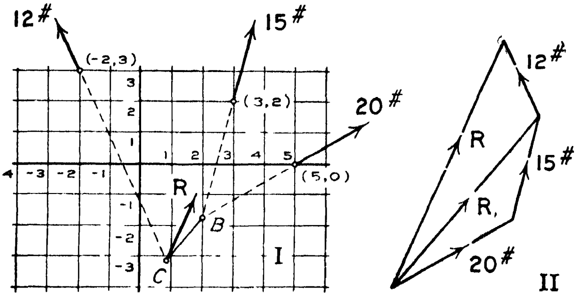

Fig. 53.

2. Given the following forces: 20 lb horizontal toward the right through the point (2,4); 25 lb at an angle of 35° to the right of the vertical through the point (3,3); 15 lb vertical through the point (1,3); and 16 lb at an angle of 45° to the left of the vertical through the point (−2,2). Find the direction and magnitude of the resultant by means of the force polygon, and find its position on the space diagram. Use 1 in. = 5 lb on the force diagram, and 1 in. = 2 units of length on the space diagram. Measure the perpendicular distance from the point (0,0) to the line of the resultant.

In order to describe completely a force in a plane three values are required: a magnitude, a direction, and a location. At first sight the student may conclude that the position or location requires two coordinates to be given, as was done for the forces in Fig. 53. But all these forces might have been located just as well by giving their intercepts on the X axis or by stating the moment of each force about some designated point. The magnitude and direction, of course, could have been alternately stated by giving two components of the force in designated directions. In any case, only three values are necessary. There are available for solutions, in all cases, three independent equations, two equations of resolution and one equation of moments. A problem of equilibrium is solvable if there are just three unknowns, when the force system is coplanar, nonparallel, and nonconcurrent.

Graphical solutions depend on closing the force polygon, in lieu of the resolution equations, and on intersections of the lines of action of the forces in the space diagram, in lieu of the moment equation. In problems involving only a few forces, intersections of the lines of forces can often be used in trigonometric or mixed types of solutions.

Example

A rectangular board (Fig. 54) is 4 ft wide and 3 ft high. A force of 20 lb, at an angle of 15° with the horizontal toward the right, is applied at the lower right corner. A force of 24 lb, at an angle of 30° to the right of the vertical, is applied at the upper right corner. A force of 16 lb at an angle of 45° to the left of the vertical is applied at the upper left corner. Find the magnitude and direction of the resultant and its distance from the lower left corner.

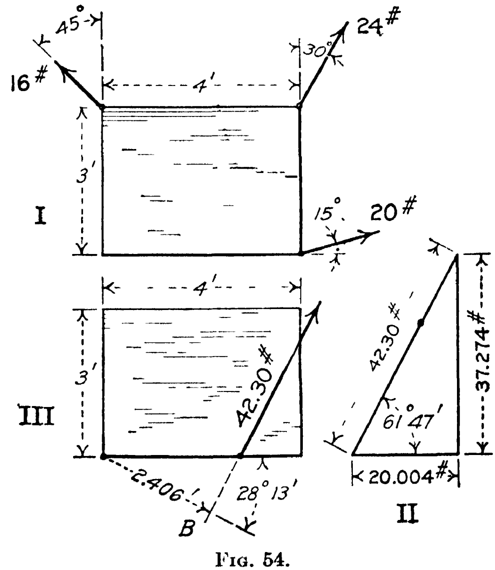

Fig. 54.

Resolving horizontally and vertically,

Force	*H component*	*V component*
20	19 318	5 176
24	12 000	20 784
16	−11 314	11 314
	20 004	37 274

The resultant is the hypotenuse of the right-angled triangle of which the base is 20.004 units and the altitude is 37.274 units. The resultant makes an angle of 61°47′ with the horizontal toward the right. Its magnitude is 42.30 lb.

To find the location of the line of the resultant force on the space diagram, moments are taken about the lower left corner of the board. Since the horizontal and vertical components of the several forces have already been computed, it is convenient to use these components in calculating the moments.

The horizontal component of the 20-lb force and the vertical component of the 16-lb force pass through the origin of moments so that the moment of each of these components is zero.

V component of 20 lb	$5\,176 \times 4 =$	$20\,704$
V component of 24 lb	$20\,784 \times 4 =$	$83\,136$
H component of 24 lb	$12\,000 \times 3 =$	$-36\,000$
H component of 16 lb	$11\,314 \times 3 =$	$33\,942$
	$M =$	$101\,782$ lb-ft

$$\frac{101\,782}{42.30} = 2.406 \text{ ft}$$

Figure 54,III locates the resultant. Since it makes an angle of 61°47′ with the horizontal, its effective arm makes an angle of 28°13′ below the horizontal toward the right. The resultant passes through the point *B* at a distance of 2 406 ft from the lower left corner of the board measured along the direction of the effective arm. If a smooth pin were placed in the board at any point along the line of the resultant, this pin could hold the board in equilibrium. The equilibrium would be stable or unstable, depending on the position of the pin in the line of action of the resultant force.

It is best to determine the sign of each moment by observing on the space diagram whether the force tends to produce rotation in the clockwise or counterclockwise sense. The student who prefers to use Formula IV should check all his results from the diagram.

The location of the resultant may be expressed in terms of its intercept on a coordinate axis. Since the origin of moments lies on the X axis, the moment of the horizontal component is zero at the intercept, and only the vertical component needs consideration. If x is the intercept,

$$37.274x = 101.782 \qquad x = 2.731 \text{ ft}$$

Since the moment is counterclockwise and V is positive upward, x is positive.

In a similar way the y intercept is found to be 5.088 ft below the lower left corner of the board.

Problems

3. In Fig. 54, find the moment of the force of 24 lb by means of its effective moment arm. *Ans.* = 24(4 cos 30° − 3 sin 30°) = 47.136 lb-ft.
4. In Prob. 1, find the magnitude, direction, and position of the resultant algebraically
Ans **H** = 16.131 lb; **V** = 35 365 lb; **R** = 38 87 lb at 24°31′ to the right of the vertical; M = 79.164 lb-ft counterclockwise, d = 2 037 ft.
5. In Prob. 4, find the X and Y intercepts of the resultant. Compare with the graphical solution.
6. In Prob. 4, compute the moment about the point (3,2) and find the distance of this point from the intersection of the resultant with the X axis.

45. Resultant of Parallel Forces in a Plane. When all the forces in a plane are parallel, a resolution equation in the direction of the forces is the only independent resolution equation, since a resolution in any other direction multiplies all forces by the same cosine and this cosine can be canceled out. A resolution in a direction perpendicular to the forces is meaningless, but there is still the independent moment equation. The deficiency of one moment equation is offset by the lack of any components of force in one direction. There are then two independent equations, and (the direction of parallelism being understood and the sense of the force being indicated by a plus or minus sign as a part of its magnitude) only two values are required to determine completely a single force in the system, following the general rule stated on page 41.

Parallel forces do not intersect. Consequently, the method of locating the resultant by intersections in graphical solutions must be modified. The modification consists of the resolution of each of the parallel forces into two components, making one or both of the components the equal collinear opposite of a component of another of the parallel forces. The equal collinear opposites cancel each other, leaving at last the whole system replaced by two nonparallel forces. In the examples below, the resultant of two parallel forces are found by this method. A more extended discussion will be given in the chapter devoted to graphical methods.

Figure 55 represents two parallel forces **P** and **Q** in the same direction. At any point A on its line, the force **P** is resolved into two components $\mathbf{S}_1$ and $\mathbf{T}_1$ at any convenient angle with each other. The lower triangle of Fig. 55,II shows this resolution. The line of the force $\mathbf{T}_1$ through A intersects the line of **Q** at B. The force **Q** at B is resolved into two components, as shown by the upper triangle of Fig. 55,II. One component is $\mathbf{T}_2$, which is equal and opposite to $\mathbf{T}_1$, the other component is S_2. The components S_1 and S_2 intersect at C on the space diagram. The resultant of S_1 and S_2 is represented by ef of the force diagram. *The resultant of two parallel forces is in the direction of the forces and is equal in magnitude to their sum.*

The moment of the resultant **R** is equal to the sum of the moments of S_1 and S_2 about any point whatever, by Varignon's theorem. T_1 and T_2 cancel each other, so that the moment of S_1 and S_2, or **R**, about any point is equal to the sum of the moments of **P** and **Q** about the same point. The moment of **R** about a point on its line of action is zero. *The sum of the moments of all the forces of the system about any point on the line of action of the resultant is zero.*

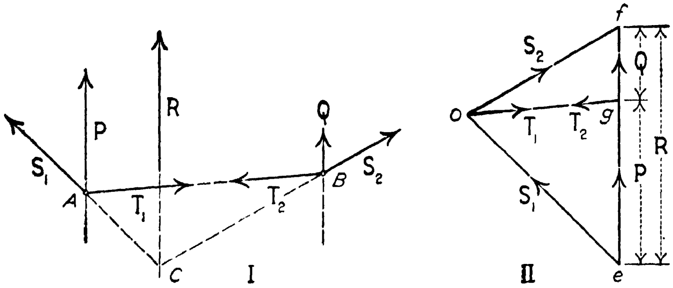

FIG. 55.

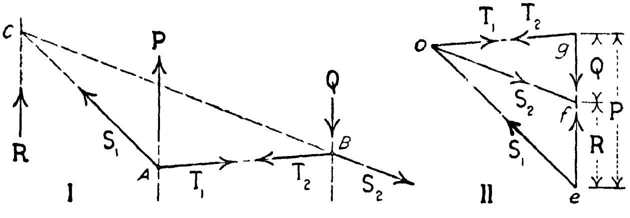

FIG. 56.

In Fig. 56, the force **Q** is opposite in direction to the force **P**. The resultant is **P** − **Q** and lies to the left of the force **P** on the space diagram when the force **Q** is on the right and is smaller than **P**.

Problems

1. Given vertical forces of 6 lb upward and 8 lb upward at a distance of 5 ft apart. Find their resultant graphically to the scale of 1 in. = 1 ft and 1 in. = 4 lb.
2. Solve Prob. 1 if the 8-lb force is downward. Use 1 in. = 2 ft on the space diagram.

In Fig. 57,I, **P** and **Q** are two parallel forces at distances x and y, respectively, from the line of their resultant. Taking moments about a point on the resultant,

$$Px - Qy = 0 \qquad Px = Qy \tag{1}$$

$$\frac{P}{Q} = \frac{y}{x} \tag{2}$$

Equation (2) is called the *law of the lever.*

It is not necessary that x and y should be perpendicular to the direction of the forces, provided they are measured along a straight line. From Fig. 57,II, the moment equation is

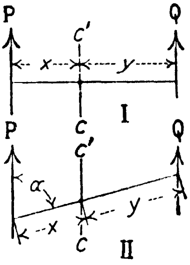

FIG. 57.

$$Px \sin \alpha = Qy \sin \alpha$$
$$Px = Qy$$

The resultant of a set of parallel forces is equal to the algebraic sum of the forces, and the moment of the resultant about any point is equal to the algebraic sum of the moments of the separate forces about that point.

Example 1

Two forces of 12 lb and 18 lb are parallel, in the same direction and 5 ft apart. Find the magnitude and position of their resultant.

Resultant = 12 + 18 = 30 lb

Taking moments about a point in the line of the force of 12 lb (Fig 58).

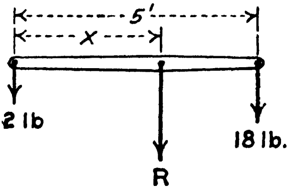

FIG. 58.

$$\begin{aligned} 12 \times 0 &= 0 \\ 18 \times 5 &= 90 \\ 30x &= 90 \\ x &= 3 \text{ ft} \end{aligned}$$

The resultant is a force of 30 lb at a distance of 3 ft from the force of 12 lb

Example 2

Find the magnitude and position of the resultant of the following vertical forces: 8 lb down at 2 ft, 11 lb down at 5 ft, 9 lb up at 7 ft, and 10 lb down at 12 ft. Solve by moments about the zero position.

Since most of the forces are downward and most of the moments are clockwise, downward force and clockwise rotation are taken as positive. It is convenient to arrange the solution in tabular form.

Force, lb	*Moment arm, ft*	*Moment, lb-ft*
8	2	16
11	5	55
−9	7	−63
10	12	120
20		$20x = 128$
		$x = 6.4$ ft

Problems

3. Solve Example 2 by moments about the 12-ft position. Call counterclockwise rotation positive.

4. Check Example 2 by moments about the resultant.

5. Find the position and magnitude of the resultant of 20 lb down at 0 ft and 12 lb up at 6 ft. Take moments about a point in the line of the 12-lb force. Check by moments about the resultant.

6. A beam 20 ft long weighs 40 lb and has its center of mass at the middle. It carries 12 lb 3 ft from the left end, 16 lb 9 ft from the left end, and 12 lb 5 ft from the right end. At what point will a single rope support this loaded beam? Check.

7. Prove the law of the lever from Fig. 55 by showing that

$$Q \times CB \sin ofg = P \times CA \sin oeg.$$

46. Equilibrium of Parallel, Coplanar Forces.

A system in equilibrium has no resultant. Consequently, if a body is in equilibrium, the (vector) sum of the forces which act on it is zero, and the moment of these forces about any axis is zero. If the system in question is one of parallel coplanar forces, it is sufficient to say that the algebraic sum of the forces (with due regard to plus and minus signs) is zero and the moment about any chosen point in the plane is zero. Once the direction of parallelism is understood, all directions are known and the unknowns are magnitudes and positions in any combination of two. There are available two independent equations, one resolution and one moment equation. The resolution equation may be replaced by a second moment equation, but the moment equation cannot be replaced by a second resolution equation.

Example

A horizontal beam is 20 ft long. It weighs 240 lb and its center of mass is at the middle of its length The beam is supported at the left end and at 4 ft from the right end. It carries a load of 120 lb 6 ft from the left end, and a load of 160 lb at the right end. Find the reaction at each support.

Take moments about the left support, in order to eliminate the reaction R_1 (Fig. 59). Call the downward force positive and the distance toward the right positive.

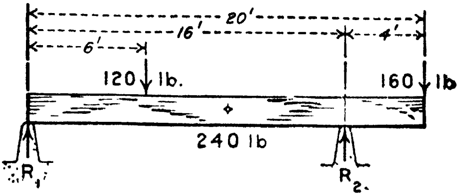

FIG. 59.

Force, lb	*Arm, ft*	*Moment, lb-ft*
240	10	2400
120	6	720
160	20	3200
520		6320

$$16R_2 = 6320$$
$$R_2 = 395 \text{ lb}$$

Take moments about the right support and call the distance toward the left positive.

Force, lb	*Arm, ft*	*Moment, lb-ft*
240	6	1440
120	10	1200
160	−4	−640
520		2000

$$16R_1 = 2000$$
$$R_1 = 125 \text{ lb}$$

Check by vertical resolutions; 395 + 125 = 520 lb.

Problems

1. A horizontal beam, with its center of mass at the middle, weighs 440 lb. The beam is 20 ft long and is supported at the right end and 4 ft from the left end. It carries 200 lb on the left end, and 360 lb 2 ft from the right end. Find the reactions of the supports and check.
2. In Prob. 1, find the distance of the resultant of the downward forces from the left support. Find the sum of the moments of the two reactions about the line of this resultant.
3. A horizontal beam, 12 ft long, weighs 60 lb and has its center of mass at the middle. The beam is hinged at the right end and is supported 8 ft from the left end. It carries a load on the left end which makes the total reaction at the support 330 lb. Find this load and the reaction at the hinge. Check.

47. Equilibrium of Nonconcurrent, Nonparallel, Coplanar Forces.

The forces which act on a rigid body may be divided into two groups. In order that equilibrium may exist, two conditions must be satisfied. These are:

1. The resultant of the forces of one group must be equal and opposite to the resultant of all the other forces.

2. The resultant of the forces of one group must lie along the same line as the resultant of all the other forces

The first condition is equivalent to the statements that the force polygon must close and the resultant force is zero This condition is expressed algebraically by two resolution equations,

$$\Sigma P \cos \alpha = 0 \qquad (1)$$

$$\Sigma P \cos \beta = 0 \qquad (2)$$

of the form as given by Eqs (1) and (3) of Art 22 Condition (1) is the same for concurrent and for nonconcurrent forces

When the force polygon closes, the force system may be divided into two groups, with any forces placed in one group and all the remaining forces in the other, and the resultants of the two groups will be equal and opposite If the forces are concurrent, the resultants of both groups must pass through the point of concurrency and so will be collinear, but if the forces are nonconcurrent these two resultants will not lie along the same line unless the sum of their moments about any point is zero From this it follows that the sum of the moments of all the forces of the system with respect to any point must be zero This condition is expressed by the equation

$$\Sigma M = 0 \qquad (3)$$

The basic equations available for the solution of a problem of coplanar, nonconcurrent forces are two resolutions and one moment The moment equation is indispensable (except by some process involving the intersections of the lines of action of the forces of the system), but one or both of the resolution equations may be replaced by additional moment equations When this is done the conditions for independent equations as shown in Table 1 must be observed

TABLE 1 CONDITIONS FOR INDEPENDENT EQUATIONS OF EQUILIBRIUM

Concurrent, coplanar forces	Nonconcurrent, coplanar forces
Two resolutions	One moment, two resolutions
One moment, one resolution *The resolution must not be perpendicular to the line joining the origin of moments to the point of application of the forces*	Two moments, one resolution *The resolution must not be perpendicular to the line joining the two origins of moment*
Two moments *The two origins of moment and the point of application of the forces must not line on the same straight line*	Three moments *The three origins of moment must not lie on the same straight line*

A resolution perpendicular to an unknown force or a moment about a point in its line of action eliminates that force. Moment about the intersection of two unknown forces eliminates both. When two unknown forces intersect, it is advisable to begin by a moment about their intersection. If the direction and magnitude of one force are unknown, although the location of one point on the line of action of this force is given, the first equation should be a moment about that point.

When nonconcurrent forces are parallel, all forces have the same direction, which is known, and one resolution equation must be omitted from the list of Table 1.

Example

A bar *AB* (Fig. 60) is 20 ft long, weighs 60 lb, and has its center of mass 8 ft from *A*. The bar is hinged at *A* and supported by a cord at *B*. The bar makes an angle of 15° above the horizontal toward the right, and the cord makes an angle of 35° to the left of the vertical. Find the tension in the cord and the direction and magnitude of the hinge reaction at *A*.

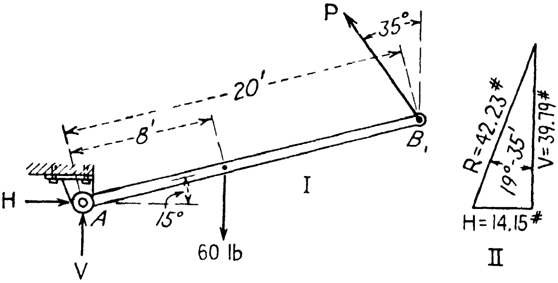

FIG. 60.

The free body is the entire bar *AB*. The forces which act on the free body are its weight, the unknown tension in the cord at *B*, and the reaction of the hinge at *A*. The hinge reaction as to both direction and magnitude is unknown.

Begin by writing a moment equation about the hinge *A*, since this eliminates two unknowns.

$$60 \times 8 \times \cos 15^\circ = P \times 20 \cos 20^\circ \qquad (1)$$
$$24 \times 0.9659 = 0.9397P$$
$$P = 24.67 \text{ lb}$$

To find the direction and magnitude of the reaction at the hinge, it is convenient to regard it as made up of a horizontal component **H** and a vertical component **V**. Resolving horizontally,

$$H = P \sin 35^\circ = 14.15 \text{ lb} \qquad (2)$$

Resolving vertically,

$$V = 60 - P \cos 35^\circ = 60 - 20.21 = 39.79 \text{ lb} \qquad (3)$$

From the force triangle (Fig. 60,II) the resultant reaction of the hinge is found to be 42.23 lb at an angle of 19°35′ to the right of the vertical.

Check all results by moments about B.

$$60 \times 12 \cos 15° = 42.23 \times 20 \cos 34°35' \qquad (4)$$

The reaction at the hinge may be found by means of components perpendicular and parallel to the bar. The component perpendicular to the bar may be represented by **N** and the other by **Q**.

(*The student will make a new sketch, showing these components, omitting* **H** *and* **V**.) The component **N** may be found by moments about **B**.

$$20N = 60 \times 12 \cos 15°$$
$$\therefore N = 36 \times 0.9659 = 34.77 \text{ lb}$$

By resolutions parallel to the bar,

$$Q = 60 \sin 15° + P \sin 20°$$
$$60 \times 0.2588 = 15.528$$
$$24.67 \times 0.3420 = \underline{\ 8.437}$$
$$Q = 23.965 \text{ lb}$$

Problems

1. Find the direction and magnitude of the resultant of **N** and **Q**, and compare with the resultant of **V** and **H** of the example above.
2. Calculate **N** of the example by resolutions perpendicular to the bar.
3. A bar similar to that of Fig. 60 makes an angle of 30° below the horizontal and is supported at B by a rope which makes an angle of 10° to the left of the vertical. The bar is 10 ft long, weighs 100 lb, and has its center of mass 4 ft from A. Find the tension in the rope by means of the effective arm. Check by Formula IV. Find the resultant at the hinge by horizontal and vertical resolutions. Check by normal and parallel components. Check **again** by moments about B.

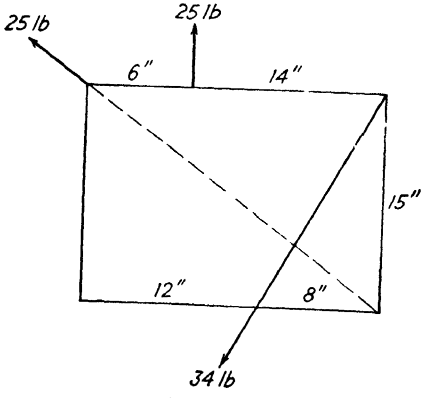

FIG. 61.

4. Figure 61 shows a rectangular board with three nonconcurrent forces acting on it in the plane of the board. Neglect the weight of the board and find the magnitude and position of the resultant of the forces. Find where the resultant cuts the lower edge of the board.

5. A uniform bar weighing 60 lb is hinged at the left end, and the right end is elevated 6 ft above the level of the left. The bar is 12 ft long and a 200-lb load is hung from the right end. A cord attached 4 ft from the right end passes directly over the hinge making an angle of 40° with the bar. Find the tension in the cord and the components of the reaction at the hinge.
6. A vertical door (Fig. 62) is 12 ft wide, 10 ft high, and weighs 360 lb. The hinges are 1 ft from the top and bottom, respectively, and are so placed that the vertical load is carried by the lower hinge. Find the direction and magnitude of each hinge reaction.
7. A uniform bar AB is 10 ft long and weighs 56 lb. Its left end rests on a horizontal floor and the right end is held 6 ft above the floor by a cord inclined toward the left. If the bar is about to slip on the floor at the left end, what is the direction of the cord and the force which it exerts if the coefficient of friction is 0.4?

Ans. 22.63 lb at an angle of 45° with the vertical.

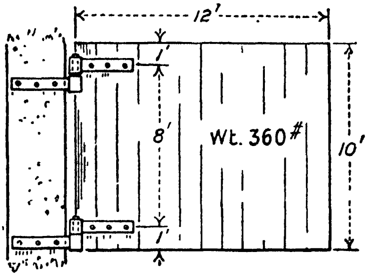

FIG. 62.

8. Solve Prob. 7 with the cord inclined toward the right.
9. A bar AB is 7 ft long, weighs 28 lb, and has its center of gravity 3 ft from the left end A. End A rests on a horizontal table and the bar is held at an angle of 35° with the horizontal by a cord attached at B and making an angle of 50° with the bar. Find the necessary coefficient of friction at A to prevent sliding.

Ans. $\mu = 0.502$.

10. Solve Prob. 9 if the cord makes an angle of 20° with the bar.
11. A ladder 30 ft long rests on a horizontal floor and leans against a smooth vertical wall, as in Fig. 63. The ladder makes an angle of 20° with the vertical. The center of gravity of the ladder and its load is 25 ft from the lower end. What must be the coefficient of moving friction at the bottom so that the ladder will not slide down after it has been started by a slight jar? Solve by one moment and two resolution equations together with Formula V. *Ans.* $\mu = 0.303$.
12. Solve Prob. 10 if the wall is not smooth but has a coefficient of friction equal to that of the horizontal floor.

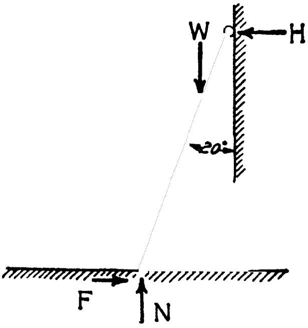

FIG. 63.

48. Stability of Parallel Force Systems. The preceding discussions and examples of equilibrium have dealt with the magnitudes and lines of action of the forces, and it has been stated that a force may be regarded as applied anywhere along its line of action or may be replaced by its equivalent in components anywhere along that line. The actual points of application of the forces on the physical free body have been seemingly unimportant. But these points are the critical determinants of the condition of stability once equilibrium is established, and by changing them to other points on the lines of action of the forces the condition of stability can be changed.

Figure 64 shows a rigid body which is supported at B and loaded at A. A second load Q acts along the line through C_2, C, and C_1. If the body is in equilibrium in this position, it is in equilibrium whether the load is applied at C, at C_1, or at C_2. The points A, B, and C lie in a straight line, as shown in Fig. 65,I. If the beam is turned through an angle θ from the horizontal position, the moment arm of the force **P** becomes $x \cos \theta$ and the moment arm of **Q** becomes $y \cos \theta$. In the original position of equilibrium

$$Px = Qy$$

Consequently,

$$Px \cos \theta = Qy \cos \theta$$

There is the same relative change in the moments of the two forces;

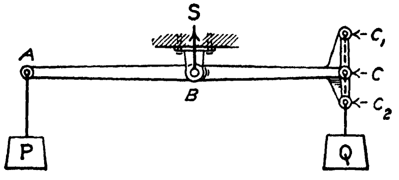

FIG. 64.

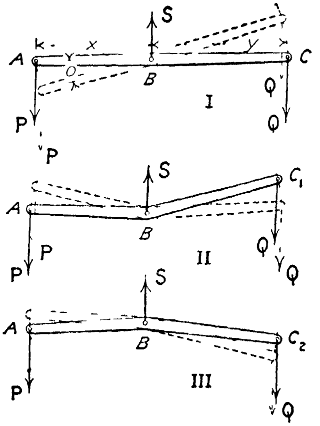

FIG. 65

therefore, the beam is in equilibrium in the new position

When a body is in equilibrium under the action of three parallel forces, if the points of application of the three forces lie in a straight line, the equilibrium is neutral.

Figure 65,II represents the case in which the load is applied at a point C_1 placed above the line through A and B. The broken lines illustrate the condition when the beam is rotated in a clockwise direction about B. The moment arm of **Q** becomes longer and the moment arm of **P** becomes shorter. After the arm BC has passed the horizontal position, the moment arm of **Q** diminishes, but less rapidly than the moment arm of **P**. If the forces **P** and **Q** in the original position produces equal and opposite moments about B, the moment of **Q** in any one of the new positions is greater than the moment of **P**, and the beam continues to rotate about B through approximately 180° to a position of *stable* equilibrium. The equilibrium of Fig. 65,II is *unstable.* If the beam has been rotated in the opposite direction from the position of equilibrium, the same condition would obtain and it would continue to rotate in that direction approximately 180° to the position of stable equilibrium.

In Fig. 65,III, the load **Q** is applied at C_2, which is below the line through A and B. When the beam is rotated slightly from this position in a clockwise direction, the moment arm of **P** shortens less than that of **Q**. The beam will return to its original position after displacement. The equilibrium is *stable.*

Example

Figure 66 represents a rectangular board in a vertical plane supported by a smooth hinge at the middle. A load of 10 lb is suspended from A_2 and a load of Q lb is suspended from C_2. The board rotates through an angle of 10° in a clockwise direction. Find the load Q.

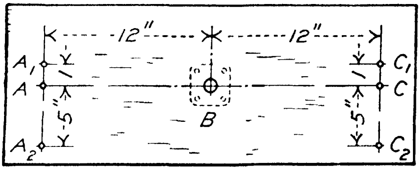

FIG. 66.

It is assumed that the center of mass is at the center of the board, so that its weight exerts no moment about the support in any position.

Moments are taken about B. (The student should make a sketch of the board after it has rotated 10° and should show the effective moment arms.) The apparent moment arms are each 13 in. long at angles of 22°37′ below AC. After the board has rotated 10°, the equilibrium equation is

$$10 \times 23 \cos 12°37' = Q \times 13 \times \cos 32°37'$$

By using the horizontal projections of the lines BA and AA_2 and BC and CC_2, the moment equation may be written

$$10(12 \cos 10° + 5 \sin 10°) = Q(12 \cos 10° - 5 \sin 10°)$$

Problems

1. In Fig. 66, the load at A_2 is 10 lb and the load at C_2 is 12 lb Find the position of equilibrium. *Ans.* The rotation is 12°19′ clockwise.
2. Solve Prob. 1 if the loads are 20 lb and 22 lb, respectively.
3. In Fig. 66, the board weighs 4 lb and its center of mass is 2 in. below the point of support. The load **P** is attached at A and the load **Q** at C. The points A, B, and C are in a horizontal straight line. Find the position of equilibrium if **P** is 10 lb and **Q** is 11 lb and there is no friction at B. *Ans.* Rotation $= \tan^{-1} \frac{3}{2} = 56°19'$.
4. Solve Prob. 3 for loads of 5 and 6 lb, respectively.
5. In Prob. 3, the loads are suspended from A_1 and C_1, which are 1 in. above A and C, respectively. Find the position of equilibrium when **P** is 1 lb and **Q** is 2 lb. *Ans.* Rotation $= \tan^{-1} 2.4 = 67°23'$.
6. Solve Prob. 5 for loads of 4 and 5 lb.

7. In Fig. 66, a load of 10 lb is applied at A_2 and a load of 11 lb at C_2. The board weighs 4 lb and its center of mass is 2 in. below B. Find the position of equilibrium. *Ans.* Rotation = 6°04′.
8. A circular disk 10 in. in diameter is pivoted on a pin 2 in. from its axis. A cord wound round the circumference carries a load of 10 lb on one end and a load of 6 lb on the other. If the disk is counterbalanced to bring the center of gravity to the pivot, what angle does the line joining the pivot to the axis of the disk make with the horizontal when the forces are in equilibrium? *Ans.* 51°19′ or −51°19′.
9. Solve Prob. 8 if the disk weighs 2 lb and has its center of gravity at the axis. *Ans.* 56°15′ or −56°15′.
10. A balance beam weighs 50 grams and its center of gravity is 0.008 in. below the central knife-edge. The knife-edges are 4 in. apart and are in the same plane. The pointer is 10 in. long. How much will the end of the pointer move when 1 mg. is placed on one pan? *Ans.* 0.10 in.

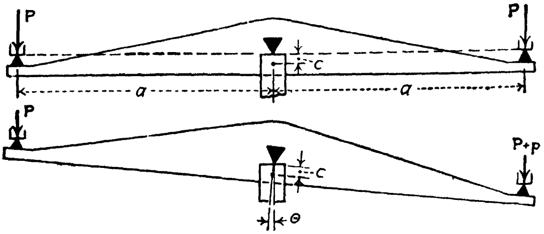

FIG. 67.

11. The center of gravity of a balance beam is frequently adjusted by means of a nut which is moved up or down on a screw at the top. (This is not shown in Fig. 67.) If the balance of Prob. 10 has a nut weighing 2 grams, how much must it be moved to double the sensibility, *i.e.*, to double the motion of the pointer due to the milligram on one pan?

49. Direction Condition of Equilibrium. When a body is in equilibrium under the action of three (apparently) nonconcurrent forces, the third force has no moment about the point of intersection of the other two, from the principle of $\Sigma M = 0$ for equilibrium. It follows, therefore, that the third force passes through the intersection of the other two and lies in the plane of the other two. This relation is sometimes called the *geometrical condition of equilibrium* but in this text it will be called simply the *direction condition of equilibrium.* The direction condition of equilibrium is the key to graphical solutions of problems of nonconcurrent forces, and it can often be used to advantage in problems like those in the preceding list, as will be illustrated in the examples below.

To make use of the direction condition of equilibrium in problems involving more than three forces, it is necessary to replace pairs of forces by their resultants (passing through their points of intersection, of course) until the system is reduced to a system of three forces. A well-developed sense of the direction condition of equilibrium assists the student greatly in making correct assumptions as to the unknown directions in any problem and generally in understanding force relations. Figure 68 shows a bar AB with a mass center at C, hinged at A and supported by a cord at B. The dotted vertical line CD is the line of action of the weight, and BD is the direction of the cord extended to the intersection D. The hinge reaction is in the direction DA. Having found the directions of the forces, the problem can be solved graphically by a force triangle or analytically by two resolution equations.

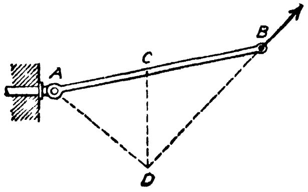

FIG. 68.

Example 1

(*Problem* 9, *Art* 47.)

Figure 69 is the same as Fig 63 with F and N replaced by their resultant R. If the ladder is about to slip, ϕ is the angle of friction and $\tan \phi = \mu$.

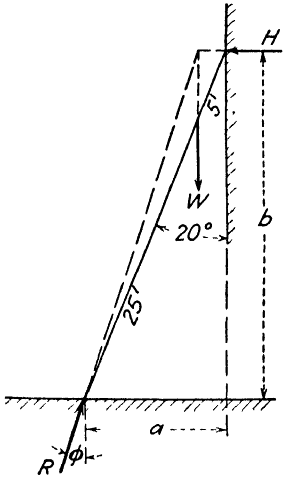

FIG. 69.

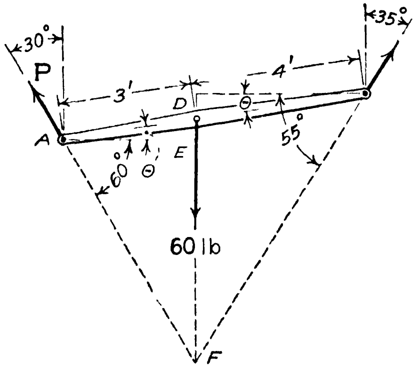

FIG 70.

$$\tan \phi = \frac{a}{b} = \frac{25 \sin 20°}{30 \cos 20°} = \frac{5 \tan 20°}{6} = 0.303$$

Example 2

A bar 7 ft long, with its center of mass 3 ft from the left end, is supported by a rope at the left end which makes an angle of 30° to the left of the vertical, and a

rope at the right end which makes an angle of 35° to the right of the vertical. **What angle does** the bar make with the horizontal? If the bar weighs 60 lb, what ***is the tension in each rope?***

From Fig. 70,

$$DF = DE + EF$$
$$4 \cos \theta \tan 55^\circ = 3 \cos \theta \tan 60^\circ + 7 \sin \theta$$
$$\tan \theta = \frac{4 \tan 55^\circ - 3 \tan 60^\circ}{7} = 0.0737$$
$$\theta = 4^\circ 13'$$

By a resolution perpendicular to the direction of the rope at the left end,

$$Q \cos 25^\circ = 60 \sin 30^\circ$$
$$Q = 30 \sec 25^\circ = 30 \times 1.1034 = 33.10 \text{ lb}$$

Check **Q** and θ by moments about the left end. Solve for **P** by a resolution equation and check.

Problems

1. A horizontal bar 5 ft long, weighing 30 lb, with its center of mass 2 ft from the left end, is supported by cords at the ends. The cord at the left end makes an angle of 30° to the left of the vertical. Find the direction of the cord at the right end. The line connecting D with B of Fig. 71 gives the direction of the force Q and makes it possible to draw the force triangle and find the unknown forces graphically. For algebraic calculations, the direction condition replaces the one moment equation which is required in the solution of a problem of nonconcurrent forces.

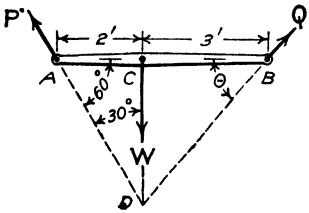

FIG. 71.

To find the angle θ from the right-angled triangles of Fig 71,

$$CD = 2 \tan 60^\circ = 3 \tan \theta$$
$$\tan \theta = \frac{2 \times 1.7321}{3} = 1.1547$$
$$\theta = 49^\circ 06'$$

This method of solution is especially valuable when only the direction is required. In practical work it is often desirable to know the direction of a force, in order to put in a support in the best position.

2. Solve Prob. 1 graphically. Draw the space diagram of Fig. 71 to the scale of 1 in. = 1 ft. Having found the direction of DB by the direction condition of equilibrium, construct the force triangle to the scale of 10 lb = 1 in. Compare with the algebraic solution of Prob. 1.

3. A bar AB, 20 ft long, weighs 60 lb and has its center of gravity 8 ft from the left end A. It is hinged at the left end, and the right end is elevated 15° above the horizontal and supported by a cord which makes an angle of 35° to the left of the vertical. Find the direction of the hinge reaction graphically by the direction condition of equilibrium. Then draw a force triangle to the scale of 20 lb = 1 in. and find the tension in the rope and the resultant hinge reaction. Compare with the example of Art. 47.

4. A ladder 20 ft long, weighing 40 lb, with its center of mass 8 ft from the lower end, rests on a smooth horizontal floor and leans against a smooth vertical wall. It is held from slipping by a horizontal force of 12 lb at the bottom. Find the position of equilibrium.

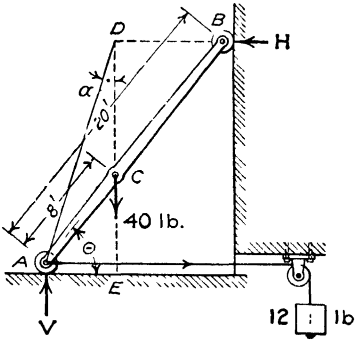

FIG. 72.

The line AD of Fig. 72 shows the direction of the resultant of the vertical reaction **V** and the horizontal pull of 12 lb at A. It also shows the direction of the resultant of the weight of 40 lb and the horizontal force **H** at their intersection D. The resultant of the horizontal force and the vertical reaction at the bottom must pass through the point D (Fig. 72), at which the vertical line through the center of the mass intersects the horizontal line through the top of the ladder. By vertical resolution, the vertical reaction at the bottom is found to be 40 lb. The resultant of 40 lb and 12 lb makes an angle with the vertical whose tangent is 0.3. From the figure,

$$DE = 20 \sin \theta = 8 \cos \theta \cot \alpha$$

$$\tan \theta = \frac{8}{0.3 \times 20} = 1.3333$$

$$\theta = 53°08'$$

50. Couples. Two equal parallel forces in opposite directions constitute a *couple*. The resultant of such a system is a zero force at an infinite distance—an intriguing mathematical concept of no practical value. A student who is fairly skilled in the art of depicting the free body and adept at analysis by moments and resolutions but who has never heard of a couple would find himself able to solve many problems involving couples; but such a student might be baffled by the problem of finding the resultant of a system when that resultant turned out to be a couple, and he would miss many opportunities for elegant solutions by manipulation of couples.

There are presented below some useful propositions having to do with couples, the arguments as to their validity, and some hints as to their usefulness in mechanics.

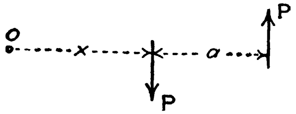

FIG. 73.

Moment of a Couple. *The moment of a couple is the product of either force multiplied by the perpendicular distance between the forces constituting the couple, and it is the same for every point in the plane.*

In Fig. 73, two forces, each of magnitude **P**, are at a distance a from each other. The distance of the downward force from any origin of moments at O is represented by x. The moment equation is

$$M = -Px + P(x + a) = Pa \tag{1}$$

A couple is the limiting case of two parallel forces in opposite directions. As shown in Art. 45 (Fig. 56), the resultant of two such forces is their difference, and the moment of the resultant about any point is the sum of the moments of the forces about that point.

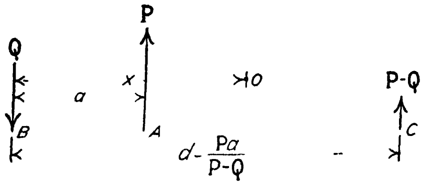

FIG 74

Figure 74 shows a force **P** upward through a point A and a smaller force **Q** downward through a point B at a distance a from the force **P**. The resultant is a force **P** − **Q** at C, which is at a distance d from the force **Q**. By moments about B,

$$(P - Q)d = Pa \quad d = \frac{Pa}{P - Q} \tag{2}$$

As **Q** is increased, the quantity **P** − **Q** becomes smaller and the distance to C increases, becoming infinity when **P** = **Q**.

When, in the solution of any problem, a couple is recognized, both of the forces of the couple may be omitted from resolution equations and the moment of the couple may be put into the moment equations regardless of the point about which moments are being taken

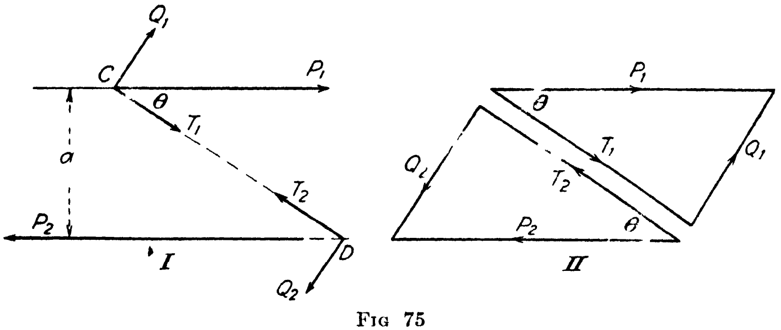

FIG 75

51. Equivalent Couples. *Any two couples in the same plane are equivalent if their moments are equal in magnitude and sign*

In Fig. 75, $\mathbf{P}_1$ and $\mathbf{P}_2$ are the equal forces of a couple with moment arm a. At any point C on $\mathbf{P}_1$, replace $\mathbf{P}_1$ by the components $\mathbf{T}_1$ in any direction θ and $\mathbf{Q}_1$ at right angles to $\mathbf{T}_1$. Extend the line of action of $\mathbf{T}_1$ to intersect the line of action of $\mathbf{P}_2$ at D. At D replace $\mathbf{P}_2$ with the component $\mathbf{T}_2$, the collinear equal and opposite of $\mathbf{T}_1$ and the component $\mathbf{Q}_2$. $\mathbf{T}_1$ and $\mathbf{T}_2$ cancel each other, leaving $\mathbf{P}_1$ and $\mathbf{P}_2$ replaced by $\mathbf{Q}_1$ and $\mathbf{Q}_2$. From the force diagram (Fig. 75,II) $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are equal and opposite with a moment arm (in Fig. 75,I) of CD.

$$CD = a \csc \theta; \qquad Q_1 = P_1 \sin \theta$$

the moment of $Q_1Q_2 = (a \sec \theta)\,(P_1 \sin \theta) = P_1a$.

In the solution of a problem of mechanics, any couple may be moved at will from any location on the free body to any other location in the plane, or it may be replaced by another couple anywhere and the forces of the replacing couple may be in any desired direction and of any desired magnitude, providing only that the replacing couple is in the same plane and has the same moment as the couple it replaces. (It will be shown later that a couple may be moved to any plane in the free body parallel to the original plane of the couple.)

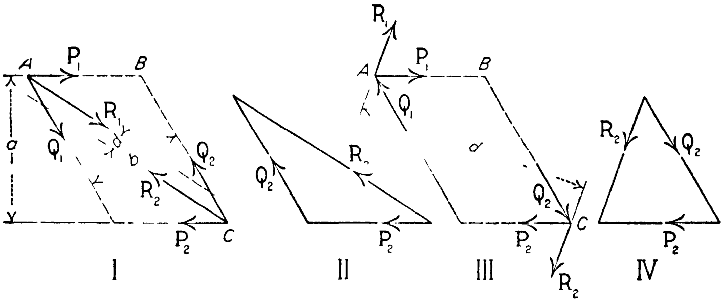

FIG. 76

52. Addition of Coplanar Couples. Figure 76,I shows two couples of unequal magnitudes and opposite sense, P_1P_2 with arm a and clockwise sense and Q_1Q_2 with arm b and counterclockwise sense. $\mathbf{P}_1$ and $\mathbf{Q}_1$ intersect at A. $\mathbf{P}_2$ and $\mathbf{Q}_2$ intersect at C. Replace $\mathbf{Q}_1$ and $\mathbf{P}_2$ at C by their resultant $\mathbf{R}_2$ and replace $\mathbf{P}_1$ and $\mathbf{Q}_1$ by their resultant $\mathbf{R}_1$. (Figure 76,II shows the vector addition.) $\mathbf{R}_1$ and $\mathbf{R}_2$ constitute a couple with moment arm d. Taking moments about C,

$$R_1d = P_1a - Q_1b$$

Figure 76,III and IV, shows the addition of two clockwise couples.

The moments of couples in the same plane are added algebraically. A system of forces which reduces to a couple cannot be further reduced. To put a couple into equilibrium requires an equal and opposite couple.

Example

Figure 77 shows a 15- by 20-in. board weighing 10 lb placed in a vertical plane and acted upon by four externally applied forces. Find the resultant of the system of forces including the weight. (The problem is the same as Prob. 4 of Art. 47 with the weight and the 36-lb horizontal force added.)

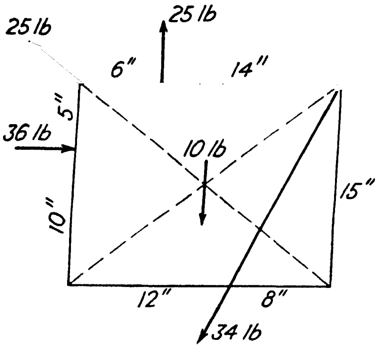

Fig. 77.

Force, lb	Component		Moment about lower right corner (counterclockwise = +), lb-in.
	Vertical	Horizontal	
25	15	−20	0
25	25	0	−350
10	−10	0	+100
34	−30	−16	+240
36	0	+36	−360
V	0		
H		0	
M			−370

The resultant is a clockwise couple of 370 lb-in. The board is not in equilibrium. It can be put in equilibrium by any counterclockwise couple in its plane if the moment of the couple is 370 lb-in.

Problems

1. Resolve the forces along one of the diagonals and take moments about the center of the board of the example.
2. Find the magnitude, direction, and position of all the forces *except* the 36-lb force. (*Set down your answer before working the problem.*)

3. Find the magnitude and direction of all the forces except the weight.
4. The board is held in position by a hinge at its lower left corner and a vertical force at the upper right corner. Find the magnitude of the force at the upper right corner and the components of the hinge force.
5. Rework Prob. 4, putting the hinge 6 in. above the lower left corner and making the force at the upper right corner horizontal.

In a coplanar system, all the couples appearing in the system may be lumped together and considered as a single couple by finding their algebraic sum (assigning, of course, opposite signs to the clockwise and counterclockwise couples).

In many problems of equilibrium the forces may be grouped to form two or more couples. Figure 78 shows a beam with loads **P** and **Q** near the ends. Neglecting the weight of the beam, the reaction $\mathbf{S} = \mathbf{P} + \mathbf{Q}$. The load **P** at the left end together with an equal amount of the reaction at the middle forms a couple of moment Pa. The load **Q** at the right end, together with the remainder of the reaction, forms an opposite couple of moment Qb. For equilibrium these couples are equal.

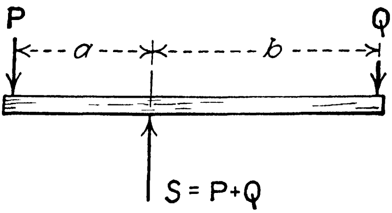

FIG. 78.

53. Relation of Forces and Couples. *A force and a couple in the same plane are equivalent to a single force, the direction and magnitude of which are the same as those of the original force. Its distance from the line of action of the original force is such that its moment about any point on that line is equivalent to the moment of the original couple.*

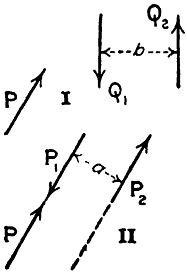

FIG. 79.

In Fig. 79,I, there is a single force **P** and a counterclockwise couple in the same plane made up of two equal forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at a distance b apart. The moment of this couple is Qb. If $Pa = Qb$, this couple may be replaced by a couple made up of two equal forces $\mathbf{P}_1$ and $\mathbf{P}_2$ at a distance a apart. By Art. 51, these forces may be placed anywhere in the plane of the couple. The force $\mathbf{P}_1$ may be placed in the line of action of the single force **P**, and the force $\mathbf{P}_2$ at a distance a from that line on the side which makes the moment counterclockwise. The force $\mathbf{P}_1$ (Fig. 79,II) balances the force **P**. The remaining force $\mathbf{P}_2$, at a distance a from the position of the force **P**, replaces the force and the couple.

Example

A vertical force of 20 lb upward is applied at the point $x = 4$ ft. A horizontal force of 5 lb toward the right is applied at $y = 2$ ft, and an equal horizontal force toward the left is applied at $y = 12$ ft. Find the location of the single force which is equivalent to this force and couple.

The moment of the couple is 50 lb-ft.

$$\frac{50}{2} = 2.5 \text{ ft}$$

The single force is 20 lb upward at a distance of 2.5 ft from the vertical line through the point $x = 4$ ft. Since the couple is counterclockwise, the distance of 2.5 ft must be measured toward the right from the line $x = 4$ ft. The resultant force lies in the line $x = 6.5$ ft.

A single force may be replaced by an equal force, in the same direction through any point in its plane, and a couple, the moment of which is the same in magnitude and direction as the moment of the original force about the point.

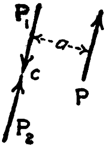

FIG. 80.

Figure 80 shows a single force **P**. It is desired to replace this force by an equal force and a couple of moment Pa. In Fig. 80, C is a point at a distance a from the line of action of the force **P**. At C are applied two opposite forces, $\mathbf{P}_1$ and $\mathbf{P}_2$, each of which is equal in magnitude to the force **P**, and along a parallel line. Since these forces balance each other, they have no effect upon the equilibrium of the body upon which they act. The force $\mathbf{P}_1$ and the force **P** form a couple of moment Pa. The force $\mathbf{P}_2$, which is equal and parallel to the original force **P**, stands alone as the single force required.

54. Shear, Moment, and Axial Forces at Sections. In finding the stresses in solids acted on by external forces, the principal topic of the science of strength of materials, a clear understanding of the relations of forces and couples becomes imperative. Figure 81,I shows a pier loaded by a force at its center. At the section S-S the unit compressive stress is simply calculated by dividing the load by the cross-section area. Figure 81,II shows the same pier loaded by a couple Pa anywhere above the section, and the stress distribution is found by a simple formula for this case, compression on one side and tension on the other. Figure 81,III shows the case where the load is off-center. This force P is replaced by a force at the center as in I, and a couple as in II, and the stress distributions for the two simpler cases are added to give the stress distribution shown in III, with a large compressive stress on one side and a small tensile stress on the other. The resultant of the distribution shown in Fig. 81,III is an *axial force* upward equal to P and a *moment* clockwise equal to Pa.

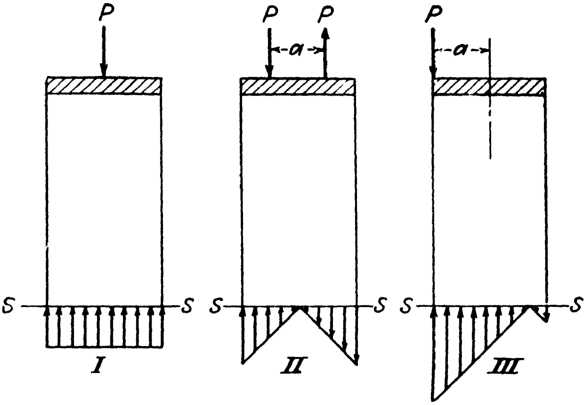

Fig. 81

Figure 82 shows a light beam 8 ft long with a single concentrated load of 400 lb located 3 ft from the right end. The *reactions,* 150 lb at the left end and 250 lb at the right end, are easily found by using the whole beam as a free body. The first step of the problem of finding the stresses anywhere on the section A-A is to take the *portion of the beam on either side of the section* as a free body and to find the force and couple which must be applied at A-A to produce equilibrium of the part under consideration. Figure 83,I shows the portion to the left of A-A and Fig. 83,II shows the portion to the right of A-A, the couple being represented by the curved arrow M and the force by the arrow V.

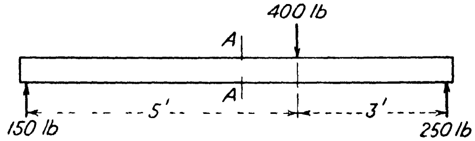

FIG. 82.

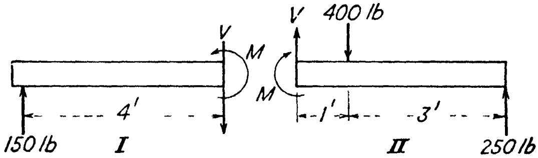

FIG. 83.

Example

Using the left portion as a free body and resolving vertically, $V = 150$ lb (downward as assumed). M can be determined by taking moments about any point on or off the free body, but it is customary to take moments about the center of the section, thus eliminating V from the computation.

$$M = 150 \text{ lb} \times 4 \text{ ft} = 150 \text{ lb-ft (counterclockwise as assumed)}$$

Problem

1. Find M and V using the portion to the right of the section (Fig. 83,II) as the free body.

COMMENT ON SIGNS. It should be noted that V comes out 150 lb *downward* when the portion to the left of the section is the free body, and it comes out 150 lb *upward* on the portion to the right. Similarly M is a *counterclockwise* couple for the left portion and a *clockwise* couple for the right portion of the beam. This is to be expected. Each of these effects is the equal and opposite reaction of its correspondent. In the terminology of beam theory, the force V at the section is called the *shear at the section* and the couple M is called the *moment at the section*. If there had been in this illustration any horizontal forces on the beam on the two sides of the section, such as would have been the case if the beam had been supported by inclined forces instead of vertical ones, then there would have been a horizontal component at the section directed toward the right on one part and toward the left on the other. This component is called the *axial force* or, sometimes, the *thrust* at the section.

Mathematicians using formal coordinates usually assume forces to the right, to the front, and upward as positive and their opposites as negative. Couples are assumed as positive when they appear as counterclockwise when viewed from the positive end of the axis looking toward the origin. Practical users of statics and dynamics normally adopt, for each free body under consideration, and even for each equation to be written from the free body, a convenient direction for positive and its opposite for negative. The unknown values are assigned assumed directions to be used in writing the equations. The actual directions are as assumed or the opposite, depending on whether the values found come out positive or negative. In beam theory a system of conventional signs is in almost universal usage. According to these conventions, the shear in the illustration is positive and the moment is positive, regardless of which side was used as a free body. If the beam is viewed from the other side, the shear by these conventions becomes negative but the moment at the section remains positive.

A set of nonconcurrent, coplanar forces can always be reduced to a *single force* or a *single couple*. The set can usually be reduced to a force and a couple in the same plane. A force and a couple can be reduced to a single force in the plane. Occasionally, the resultant force is zero and the resultant couple is not zero. This resultant couple cannot be reduced to a single force at a finite distance.

When a system to be equilibrated can be reduced to a couple, the problem is more readily solved if this fact is recognized, but it can be solved in the usual manner if it is not. The equilibrating system in this case will be a couple, or have a couple as its resultant.

When a system of forces is required to be equilibrated by a force at a given point, this force will have the same magnitude and direction as the simple single-force equilibrant of the system, but in addition to the force there will have to be applied a couple equal and opposite to the moment of all the forces of the system about the point where the equilibrating force is required to be put. (The word equilibrant is often thought of as meaning a single force, but generally it means the system required to put another system in equilibrium.)

Problems

2. A force of 12 lb vertically upward acts through the point $x = 2$ ft. A counterclockwise moment of 60 lb-ft is applied to the same body. Find the single force which is equivalent to the force and the couple.

Ans. 12 lb upward through $x = 7$ ft.

3. A force of 20 lb acts horizontally to the right through $y = 5$ ft. Replace this force by an equal force through $y = 0$ and a couple.

Ans. 20 lb through $y = 0$ and a clockwise couple of 100 lb-ft.

4. Find the resultant force or couple from 20 lb down at 0 ft, 15 lb up at 5 ft, 12 lb down at 8 ft, and 17 lb up at 10 ft. Take moments about 0 ft. Check by moments about 2 ft. *Ans.* A counterclockwise couple of 149 lb-ft.

5. Find the resultant force at (0,0) and the couple from 10 lb up at (0,0), 12 lb up at (2,2), 15 lb down at (5,2), 20 lb horizontal toward the right at $(3,y_1)$, and 20 lb horizontal toward the left at $(5,y_1+5)$.

Ans. Force = 7 lb up; couple = 49 units counterclockwise.

6. A light beam 12 ft long is supported by vertical forces at the ends and loaded with two 400-lb loads, one of them 4 ft from the left end and the other 4 ft from the right. Find the shear and moment at a section 5 ft from the left end. Check by using the other portion as a free body.

Ans. $V = 0$; $M = 1600$ lb-ft, acting counterclockwise on the portion to the left of the section.

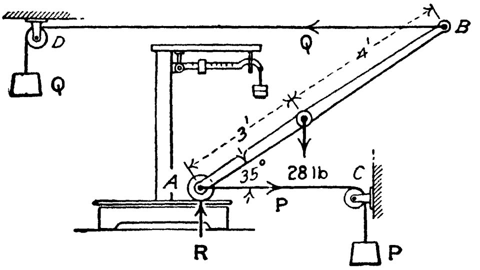

FIG. 84.

7. The bar AB (Fig. 84) is 7 ft long, weighs 28 lb, and has its center of mass 3 ft from A. The end A is provided with a cylindrical roller which allows it to move on a surface with little friction. The bar is placed with the end A on a horizontal platform. It is supported by a cord *BD* attached at *B* and is held at *A* and by a second cord *AC*. Find the tension in each cord and the reaction of the platform when the bar makes an angle of 35° with the horizontal and both cords are horizontal. *Ans.* $P = Q = 17.14$ lb; $R = 28$ lb.

8. Solve Prob. 7 with the angle changed from 35 to 45°.

9. Visualize the lower hinge reaction in Fig. 62 as separate components, and solve the problem by equating the two couples.

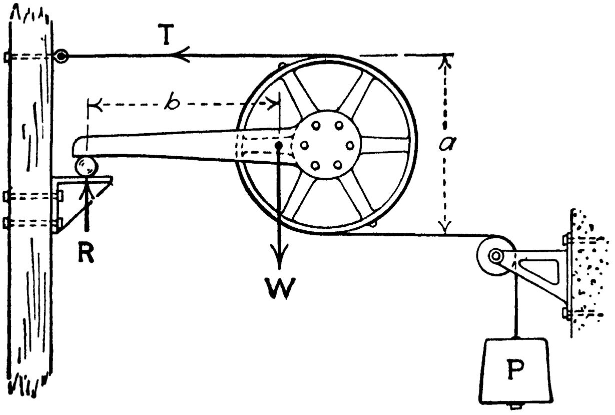

Fig. 85.

10. Write three equations from Fig. 85. If the cord at the top of the figure were shortened a little, would the wheel move up, or down?

55. Questions

1. How many independent equations may be written for a problem of nonconcurrent, nonparallel, coplanar forces? How many equations may be written if the forces are parallel?

2. What precaution must be taken when three moment equations are used? Why? What is the similar case for concurrent forces? What precaution must be taken when two moment equations and one resolution equation are written?

3. By what two methods may one unknown be eliminated? How may two unknowns be eliminated? When is it possible to eliminate two unknowns in writing a resolution equation?

4. Can a set of nonconcurrent, coplanar forces in equilibrium be transformed into a set of couples in equilibrium by resolving some of the forces into components?

5. Can a system of three forces in equilibrium be really nonconcurrent? Noncoplanar?

6. If a force in a system is moved to a new line of action parallel to its old line, what must be done to compensate for the move?

7. How can a system of parallel forces be transformed into an equivalent system of nonparallel forces?

8. Can a system of nonparallel forces be transformed into an equivalent system of parallel forces?

56. Miscellaneous Problems

1. In Fig. 60, the force at B is 20 lb. What angle does the bar make with the horizontal if the other data are unchanged? Find the components of the hinge reaction. Is it necessary to know the angle of the bar to find these components?
2. In Fig. 60, the tension at B is 20 lb. The bar remains at an angle of 15° with the horizontal but the position of the load is changed. How far is the load from A? Find the components of the hinge reaction.
3. Solve Prob. 2 graphically for all unknowns.
4. A rectangular board, 30 by 16 in., weighs 12 lb and has its center of mass at the center. The board is in a vertical plane with the 30-in. edges horizontal. A force of 6 lb perpendicular to the diagonal acts upward to the left of the vertical at the lower left corner. A force of 6 lb perpendicular to the diagonal acts downward to the right of the vertical at the upper right corner. Find the direction, magnitude, and location of a single force which can hold the board in equilibrium.
5. Figure 86 shows a light platform with one leg resting on a roof with a slope of 5/12 and the other leg extending over the peak of the roof. If the coefficient of friction at both points of contact is 0.25, what is the limiting value of x, the horizontal distance from the peak of the roof to the center of gravity of the platform and its loads? *Ans.* 3.95 ft.

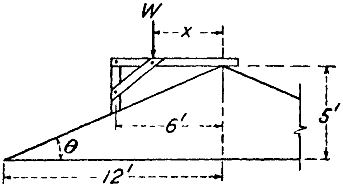

FIG. 86.

6. Solve Prob. 5 in general by letting 5 ft $= h$, 12 ft $= b$ (or use the angle θ), and 6 ft $= c$, with coefficient of friction μ. Check your solution by substituting the values from Prob. 5.
7. Solve Prob. 5 graphically.
8. A horizontal beam AB is 12 ft long, weighs 40 lb, and is hinged at the left end A. A rope runs vertically upward from the right end B, passes over two smooth pulleys and supports a mass of 24 lb on the end. The descending portion of the rope is vertical at a distance of 4 ft to the left of the ascending portion. The rope is of such length that the 24-lb mass rests on the beam. Find the tension in the rope.
9. In Fig. 84, let the force Q be 15 lb and let AB make an unknown angle θ with the horizontal. By means of the force polygon and the direction condition of equilibrium, find θ. *Ans.* $\tan\theta = 0.8$.

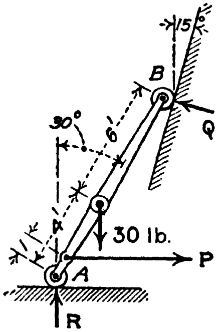

FIG. 87.

10. Find the unknown direction in Fig. 88 by an algebraic solution based on direction condition of equilibrium.
11. A bar AB is 10 ft long, weighs 30 lb, and has its center of mass 4 ft from A. The end A rests on a horizontal floor and the end B rests against a vertical wall. Both ends are provided with frictionless rollers. The bar is held by a horizontal cord attached 1 ft from A. Find the tension in the cord and the

reactions at the floor and at the wall when the bar makes an angle of 30° with the vertical.

12. Solve Prob. 11 if the wall makes an angle of 15° with the vertical, away from the bar, as shown by Fig. 87.

In order to eliminate two unknowns, it is advisable to take moments about the intersection of the vertical line through A with the line of the cord. It is best to use the vertical and horizontal components of **Q** in this moment equation.

A part of the solution is

$$
\begin{aligned}
30 \times 4 \sin 30^\circ &= Q(9 \cos 30^\circ \cos 15^\circ + 10 \sin 30^\circ \sin 15^\circ) \\
Q &= 6.800 \text{ lb} \\
P &= Q \cos 15^\circ = 6.569 \text{ lb}
\end{aligned}
$$

The student will verify these results and check by moments about A. He should find **R** by vertical resolution and check by moments about B.

13. The bar of Prob. 11 is placed upon two inclined planes (Fig. 88). The end A rests on a plane which makes an angle of 65° to the left of the vertical, and the end B rests on a plane which makes an angle of 50° to the right of the vertical. Find the reaction of each plane and the angle which the bar makes with the horizontal. Compute the reactions by resolving parallel to the inclined planes. Then find the angle by moments about A. It is best to use the components of the force **P** in the moment equation.

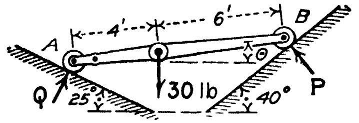

FIG. 88.

$$
\begin{aligned}
30 \times 4 \cos \theta &= P \sin 40^\circ \times 10 \sin \theta + P \cos 40^\circ \times 10 \cos \theta \\
12 &= P(\sin 40^\circ \tan \theta + \cos 40^\circ)
\end{aligned}
$$

Check by moments about B. *Ans.* $\theta = 8°07'$.

14. A beam AB of length l is fastened to a semicircular disk of radius r. The center of curvature of the disk coincides with the right end B of the beam. A load P is suspended from A and a load Q is hung on a cord which is tangent to the circumference of the disk. The body is supported on a pivot at C on the straight line which joins A to B. Find the angle which the line AB makes with the horizontal if $AB = 16$ in., $AC = 10$ in., $r = 5$ in., $P = 12$ lb, $Q = 10$ lb, and the center of mass is at the support. *Ans.* ±33°35'.

15. Solve Prob. 14 if the mass of the body is 4 lb and the center of gravity is 2 in. below C when AB is horizontal.

Ans. Left end elevated 41°54' for stable equilibrium. Left end depressed 26°43' for unstable equilibrium.

16. Solve Prob. 14 if the pivot is 1 in. above the line AB. The beam weighs 4 lb and has its center of gravity 3 in. below the pivot. All the other data remain unchanged.

17. A ladder 24 ft long with its center of gravity 16 ft from the lower end rests on a horizontal floor for which the coefficient of friction is 0.3 and leans against a smooth vertical wall. Find the maximum angle with the vertical. Solve trigonometrically by the direction condition of equilibrium and the angle of friction. (Fig. 89.)

18. Solve Prob. 17 with a coefficient of friction of 0.03 at the floor and 0.02 at the wall. (Fig. 90.) *Ans.* $\theta = 24°53'$.

19. A ladder stands on a horizontal floor and makes an angle of 30 with a vertical wall. If the coefficient of friction at floor and wall is 0.3, how far up the ladder can the center of gravity (of ladder and load) go before slipping impends? Solve graphically. A triangle described in a semicircle is a right triangle (Euclid).

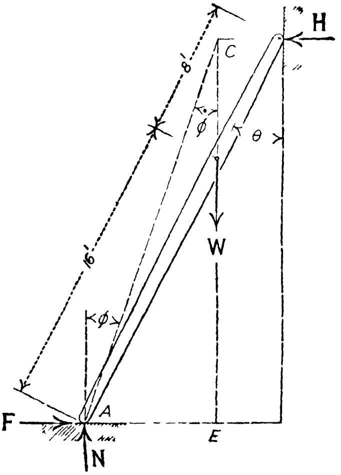

FIG. 89.

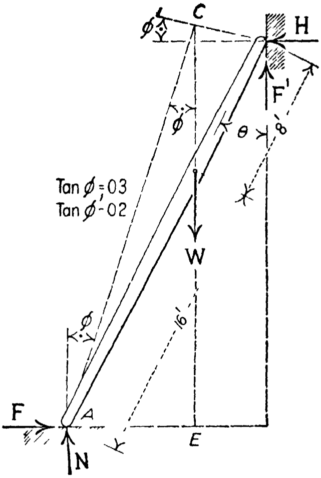

FIG. 90.

20. Solve Prob. 12 graphically. First use the intersection of R and P and the intersection of 30 lb and Q. Then solve again, using the intersection of 30 lb and P and the intersection of R and Q.

When the position of the free body is given, as in Prob. 1 of Art. 49 and the problems above, the graphical solution is convenient. When the position of the free body is not given, it often is necessary to combine the direction condition with algebraic work based on the geometry of the space diagram.

21. Solve Prob. 7 of Art. 54 graphically.

22. A slender uniform rod is placed inside a hemispherical bowl. If the rod has a length equal to the radius of the bowl and can be made to stand at an angle of 30° with the horizontal, what is the coefficient of friction between the ends of the rod and the bowl?

23. A slender uniform rod, equal in length to the diameter of the bowl of Prob. 22, can be made to stand with one end at the bottom of the bowl. What is the coefficient of friction between the rod and the bowl, assuming it to be the same at the end of the rod and the point where the rod rests on the edge of the bowl? Assume that the upper edge of the bowl is in a horizontal plane. *Ans.* 0.42. Check graphically.

24. A uniform rectangular block is pulled up a 30° inclined plane where the coefficient of friction is 0.4 by a cord attached to the upper rear corner and making an angle of 20° with the plane. The block has a length a in contact with the plane and a height b. Find the minimum ratio of a to b to keep the block from turning over rather than sliding up the plane. *Ans.* $a = 2.98b$.

25. Solve Prob. 24 with the cord attached to the upper front corner. Check graphically. (Problems 24 and 25 are easy to check graphically but require some ingenuity for a direct graphical solution.)

26. A slender uniform bar 10 ft long rests on a horizontal surface and is to be turned into the vertical position by a force always perpendicular to the bar and 2 ft from one end. What must be the coefficient of friction between the end of the bar and the surface to make this possible without providing a stop at the lower end to prevent sliding?

27. Generalize Prob. 26 by letting a equal half the length of the bar and b equal the distance from the middle of the bar to the force.

$$\textit{Ans.}\ \mu = \frac{a\cos\theta}{a\sin\theta + b\csc\theta},\ \text{where}\ \theta = \sin^{-1}\sqrt{\frac{b}{a+2b}}.$$

CHAPTER 5

CONNECTED BODIES

57. Connected Bodies. In the preceding chapters the force systems considered were acting on discrete rigid bodies, but the problems of importance in the practical application of mechanics are generally more complex then those so far treated and usually involve two or more rigid bodies connected together by cords, links, pins, bars, or other bodies. It then becomes necessary to isolate the separate parts of the whole structure or machine and to treat each part so isolated as a free body under the action of a system of forces, some of which come through the connecting members from the neighboring parts. For example, consider Prob. 7 of Art. 54. This problem dealt with the equilibrium of the bar AB of Fig. 84. Equilibrium of the bar was assumed and the numerical values of P, Q, and R were required. But there were several other bodies in the picture such as the mass Q, the pulley D, the mass P, the pulley C, and the scales by which the reaction R was supposed to be determined experimentally. If the bar AB is in equilibrium, it can be assumed that each of these other bodies is in equilibrium. The cord at A pulls, it was found, 17.14 pounds *to the right* at A, but it pulls 17.14 pounds *to the left* at the top of pulley C. Moments about the smooth pin at the center of C would show that the vertical part of the cord must pull *down* 17.14 pounds at the right side of C. Two resolution equations would show that the smooth pin at the center of C exerts a force of 24.24 pounds at an angle of 45° with the horizontal. The vertical cord pulls upward on the mass P with a force of 17.14 pounds and gravity pulls downward on this mass 17.14 pounds, the *weight* of the mass. The platform pushes *up* on the roller at A 28 pounds, and the roller at A pushes *down* on the platform 28 pounds. This pair of 28-pound forces constitutes an example of the action and reaction which are stated to be equal and opposite in Newton's third law of motion. The problem of the equilibrium of the pulley C is relatively simple, and it was assumed that the student would not stop to consider it. The problem of the equilibrium of the mass P is still simpler but is by no means trivial. It can be classified as a problem of coplanar, concurrent, parallel forces—a system having only one independent equation. The little platform scale shown in the sketch has a rigid frame integral with a vertical hollow column and horizontal beam arm, and a platform mounted by knife-edge bearings on at least two horizontal beams, one resting on the frame and on the other beam and the other resting on the frame and on a vertical hanger in the column. The vertical hanger is suspended from a knife-edge at the back end of the graduated beam, and the graduated beam is pivoted in a hanger from the beam arm and carries the movable poise and the swinging pan. The problem of determining the force R by the size and positions of the poise and the weights on the pan obviously requires the consideration of the equilibrium of a number of separate but connected pieces, if it is to be done analytically.

Statics problems involving connected bodies require the treatment of some or all of the parts as separate free bodies and, usually, the consideration of the whole group as a free body in equilibrium under the action of all the outside forces. Each part is in equilibrium under the action of the outside forces which are applied directly to it and the forces which are exerted on it by the adjacent parts. For each element used as a free body, the student should draw a clear free-body diagram, showing all that is known and indicating what is unknown.

Isolate the Free Body

The unknown genius who first conceived the idea of isolating the free body deserves a *monumentum aere perennium.*

What was said about friction and smooth surfaces, smooth pulleys and hinges, flexible cords, smooth rollers, rigid bodies, and gravity forces in Chap. 3 must now be remembered. A cord exerts a tensile force in its own direction. A light (assumed weightless) link with smooth hinges at its ends can exert a force only in its own direction, but this force may be either tension or compression. The same may be said for any rigid body with forces applied at only two points (and no couples). Such bodies or members of a structure are called *two-force members.* If a bar or other body has forces (weight included) applied at more than two points, it is called a *multi-force member* and the forces which it exerts on the parts to which it is connected are generally *not in the direction of its own length.* The bar of Fig. 87 is an example. The force which it exerts on the wall at B is the equal and opposite of **Q**, and it is not even approximately in the direction of the bar. Forces which one part of a free body exerts on other parts of the same free body are called *internal forces* and should always be omitted in writing moment and resolution equations for that free body. The action of one free body on another and the equal and opposite reaction of the second on the first must always be designated by the *same symbol in algebraic equations*, but the opposite directions must be shown consistently on the two free bodies on which they act.

The basic principles of statics are truly simple, and they can be expressed in the simple rules of resolution and composition of forces and of the writing of moment and resolution equations. These techniques are necessary to the practical solution of problems and must be mastered, but they will avail the student of mechanics little unless he is also skilled in the art of depicting the free body and all that pertains thereto. This art cannot be acquired by rule but comes only from the study and solution of a series of problems of increasing difficulty.

Example 1

One end of a beam (Fig. 91) rests on a fixed support and the other end is lifted by a 10° wedge. The beam weighs 800 lb and has its center of mass at the middle. The lower face of the wedge is horizontal on a horizontal support. The coefficient of friction is 0.3 at the upper surface of the wedge and 0.2 at the lower surface. Find the horizontal force required to push the wedge under the beam.

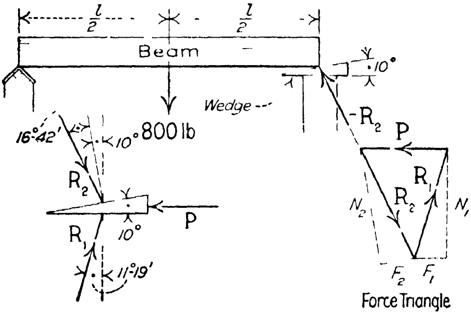

FIG. 91.

The reaction at the upper surface makes an angle of 16°42′ with the normal, or 26°42′ with the vertical. Taking the beam as the free body, the vertical component of $-R_2$ is found to be 400 lb. The horizontal component of R_2 pushing the wedge toward the right is 400 tan 26°42′ = 400 × 0.05295 = 201.2 lb. The horizontal resistance at the bottom of the wedge is 80 lb.

$$P = 281.2 \text{ lb}$$

Problems

1. What must be the coefficient of friction at the left end of the beam of Example 1 to keep it from sliding on the support? *Ans.* $\mu = 0.503$.
2. What horizontal force toward the right will draw the wedge of Example 1 from under the beam? *Ans.* Pull = 127 lb.

3. If the coefficient of friction at the upper surface of the wedge of Example 1 is 0.1 and at the lower surface is 0.05, is it possible to lift the beam a considerable distance by striking the wedge a succession of blows with a sledge hammer?

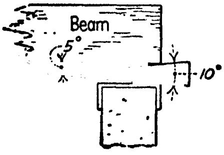

FIG. 92.

4. Solve Example 1 and Probs. 1 and 2 if the right end of the beam and the right edge of the support are in the same vertical plane, and the surfaces of the wedge each make an angle of 5° with the horizontal as in Fig. 92.

Ans. $P = 159.2 + 117.1 = 276.3$ lb; $\mu = 0.398$; pull $= 127.1$ lb.

5. In Fig. 91, where should the horizontal force P be applied in order to keep the lower face of the wedge horizontal?

Example 2

Block A in Fig. 93 is held on the inclined wall by a light rod hinged to A at its upper end and resting on the corner of block B at its lower end. The rod makes an angle of 40° with the horizontal and lies in a vertical plane through the centers of A and B. The coefficient of friction is 0.6 between the blocks and the planes on which they rest. What is the minimum weight of block B for equilibrium if block A weighs 200 lb?

The rod is hinged to A. Resting on the corner of B is equivalent to a hinge. The rod is a two-force member and so exerts a force in its own direction. Let this force be P. Assume that the proportions of block A are such that it will not turn about the hinge, *i.e.*, that the resultant of its weight and the forces exerted on it by the inclined plane can pass through the hinge. Complete the free-body diagram for block A, showing all the forces known and unknown which act on it (Fig. 94).

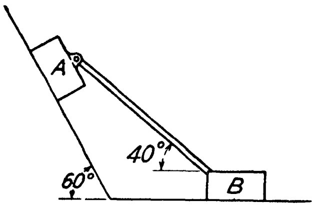

FIG. 93.

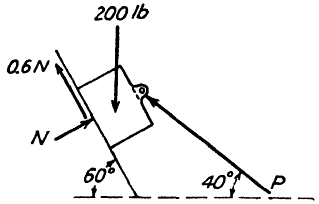

FIG. 94.

All directions are known. After replacing the friction force with $0.6N$ (it being assumed that slipping is impending) there are two unknowns, P and N. Choose a convenient pair of axes and solve by resolutions, or lay out a force diagram and scale the unknowns.

Resolving on an axis perpendicular to the force P,

$$200 \cos 40^\circ = N \cos 20^\circ + 0.6N \cos 70^\circ$$

whence

$$153.2 = 1.145N \qquad N = 133.8 \text{ lb}$$

Resolving on an axis parallel to the inclined plane,

$$P \cos 20^\circ + (0.6)133.8 = 200 \cos 30^\circ$$

whence

$$P = 98.8 \text{ lb}$$

Complete the free-body diagram for block B. Since the rod pushed *upward* on block A, it must act *downward* on B. To avoid confusion in algebraic equations, another symbol, M, is used for the normal force which the floor exerts on block B. There are now two unknowns in this free-body diagram, M and the weight of the block B. The free body is shown in Fig. 95. (It is obvious that these two free-body diagrams could have been finished just as well in Fig. 93, in which case there would be two arrows shown acting along the line of the rod, one upward at the upper end symbolizing the force the rod exerts on A and one in the opposite direction showing the action at the other end on B. The two diagrams are shown separately here for greater clarity.)

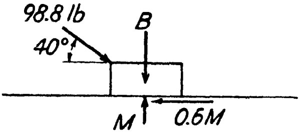

FIG. 95.

Since slipping is impending, the friction force is taken as $0.6M$. Resolving horizontally,

$$98.8 \cos 40° = 0.6M$$
$$M = 126.1 \text{ lb}$$

Resolving vertically,

$$98.8 \cos 50° + B = M = 126.1$$

whence

$$B = 62.6 \text{ lb}$$

To check the problem, put the values found for the two normal forces and their corresponding friction forces and the value B back into Fig. 93, completing the free-body diagram for the two connected bodies. The force P will not appear in the solution, as it is an *internal force*. Resolving vertically,

$$200 + 62.6 = 126.1 + 80.3 \cos 30° + 133.8 \cos 60°$$
$$262.6 = 262.5$$

or, to the accuracy maintained,

$$263 = 263$$

Resolving horizontally,

$$75.7 + 80.3 \cos 60° = 133.8 \cos 30°$$

Problems

6. The block on the plane in Example 2 has the following dimensions: The base of the block in contact with the plane is 12 in. long. The perpendicular bisector of the base passes through the center of gravity and through the hinge. The hinge is 12 in. from the base and the center of gravity is 6 in. from the base. Will the block remain in contact with the plane without rotating, as assumed? That is, does the resultant of the normal pressure act on the base of the block, or does it fall outside the base?

   Ans. The center of pressure is less than 0.6 in. from the center of the base, so the condition is as assumed.

7. A 600-lb body rests on a horizontal floor near a vertical wall and a wedge weighing 100 lb is inserted between the wall and the body, with one face of the wedge against the wall and the other face against the 600-lb body. If the angle between the faces of the wedge is 25° and the coefficient of friction between all contact surfaces is 0.231, what vertical force on top of the wedge is required to move the body?
8. Would it be possible to push the 600-lb body in Prob. 7 toward the wall and move the 100-lb wedge upward by applying a horizontal force to the 600-lb body? If not, to what value would the coefficient of friction have to be reduced to make this possible?
9. In Example 2, what would be the minimum angle between the rod and the horizontal if the block B is of negligible weight? *Ans.* 59°2′. (The rod must lie within the cone of friction.)
10. In Example 2, what would be the minimum angle between the rod and the horizontal if the block B weighs 100 lb?

Example 3

Find the force P, parallel to the plane, required to push the 20-lb rectangular block and the cylinder in front of it up the 20° inclined plane if the coefficient of friction at all contact surfaces is 0.3 (Fig. 96).

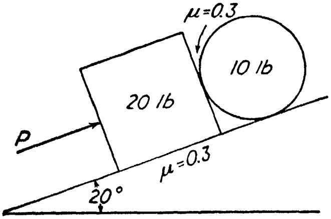

FIG. 96.

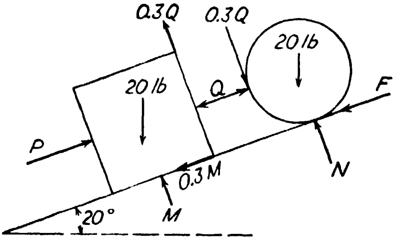

FIG. 97.

Figure 97 shows the two bodies drawn separately to show the intermediate pressure Q between them and the intermediate friction force $0.3Q$ acting mutually on the two bodies. Other forces are added to complete the two free-body diagrams. The friction force F under the cylinder is not taken as $0.3N$ because it is assumed that the cylinder will roll and not slip on the plane. It is assumed that the block will slide on the plane and not tip up at its front. The cylinder presents a problem of nonconcurrent, coplanar forces in equilibrium with three unknowns. Let the radius be r and take moments about the point of contact with the plane. Then $Q = 4.9$ lb. Resolve perpendicular to the plane and find $N = 10.37$ lb. Moments about the center or another resolution will determine F. If F turns out to be less than $0.3N$, the assumption of rolling rather than sliding is justified. Using the value of Q already found, solve for P and M with the block as the free body. Check using the two bodies together as the free body. •

Problem

11. Solve Example 3 if the front face of the block is made vertical.

58. Number and Combinations of Unknowns. Figure 98 shows a 40-pound mass suspended from a point A which is held by a horizontal force to the left and a force T at an unknown angle. Assuming the pulley to be free from friction, the horizontal force at A is obviously 10 pounds, leaving two unknowns at A. These can be solved, and the values found for magnitude and direction can be used in the solution of the problem of finding P and Q acting at B. In this system of two point-bodies connected by a two-force member, there are four unknowns, but only two of them appear at A. *In problems of this kind, begin at a point where there are only two unknowns,* carrying forward as knowns at the next free body the answers found for the first.

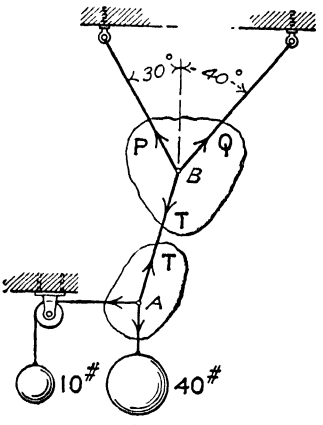

FIG. 98.

Problem

1. For the situation shown in Fig. 98, show that $T = 41.23$ lb at an angle of 14°02′ with the vertical, $P = 19.21$ lb, and

$$Q = 30.49 \text{ lb}$$

The figure might have been changed by attaching the horizontal cord from A to an eyelet, putting the little pulley at the upper end of cord Q, and suspending a mass so that Q would be known. Then if Q were made 30.49 pounds the horizontal force at A would come out 10 pounds and the problem would have the same answers as before. In this case, the problem would still present four unknowns but neither of the concurrent, coplanar problems, alone, would be solvable. The total number of unknowns equals the total number of possible independent equations.

In problems of this kind, isolate the separate free bodies and write for each its complement of equations, with due care as to the consistency of directions assumed for the internal forces. It would be absurd to assume that the cord connecting A and B in this figure pulls upward on A and also pushes upward on B without any external forces applied to it along the way. If it could do this, perpetual motion would be easy. It would not be entirely absurd to assume that the cord pushes downward on A and pushes upward on B, and if this had been done in the problem above, T would have come out with a negative sign and would still be the correct value.

Problem

2. Suppose in Fig. 98 that the horizontal cord to the left from A is attached to the support and the 10-lb mass hung from A, that the cord Q is passed through the smooth eyelet, and that the 40 lb mass is hung from this cord. Find the unknowns P, T, θ, and the unknown horizontal force at A.
Ans. $P = -23.83$ lb. (A cord could not be used; a rod or link would be necessary.) $\theta = 75°7'$ with vertical.

59. A Rigid Frame.

The simplest form of a rigid frame is made of three bars, which are hinge connected at the ends to form a triangle, as in Fig. 99. To change an angle of a triangular frame, one or more of the bars must be shortened or lengthened.

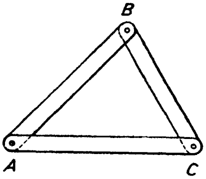

FIG. 99.

A hinged quadrilateral, as shown in Fig. 100, is not rigid. If the frame of Fig. 100,I were supported at A and C and a load applied at B, the angles at A and D would become smaller and the angles at B and C would become larger. The quadrilateral $A'B'D'C'$ of Fig. 100,II is one of the possible forms which the frame would take as it collapsed. If A and D were connected by a fifth bar, as in Fig. 100,III, the frame would be rigid. It would then consist of two triangular frames with the bar AD common to both. A connection from B to C would also render the frame rigid. A chain or flexible rod from A to D would support the frame if the applied loads tended to increase the angles at B and C but would be useless if the forces were reversed.

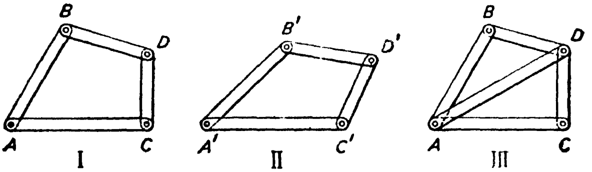

FIG. 100.

If the quadrilateral is braced by two diagonals, both of which can resist compression as well as tension, one of these is said to be *redundant* and the frame is *statically indeterminate.* A *statically determinate* frame is one for which the stresses in each member caused by a given load can be computed by the methods of statics. The computation of the stresses in a *statically indeterminate frame* involves the cross section, length, and elasticity of each member and requires a mastery of strength of materials. Only statically determinate frames are considered in this book.

Figure 101 shows a rigid frame of three light bars in a vertical plane. The bars are hinged together with frictionless pins and the frame is hinged to an outside support at one lower corner and held by a tension member attached to the upper corner. The other lower corner supports a load of 120 pounds. *All the outside forces are applied at the joints,* and the bars are considered weightless The bars, therefore, are *two-force members,* and the force exerted on any pin by any bar is in the direction of that bar. The force exerted by the outside support at the lower left is in an unknown direction, but, for convenience, it is represented in the figure as an unknown horizontal **H** and an unknown vertical **V.**

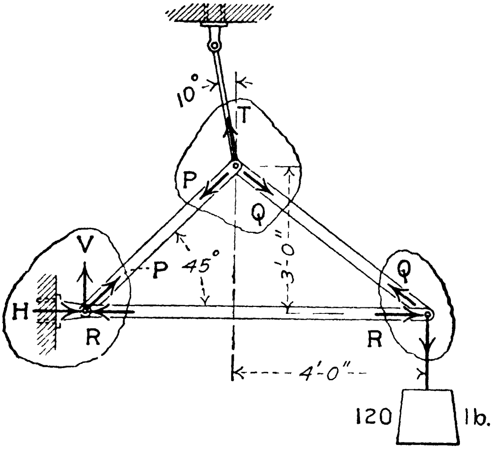

FIG. 101

The problem of analysis here is similar to the one presented in Fig. 98. In that problem the two knots at *A* and *B* were taken in order as free bodies. In this one the pins should be taken as the free bodies in the order of lower right, upper, lower left. From the solution of these three concurrent, coplanar problems will come a value for each of the outside forces, and these can then be checked by taking the whole triangle as a free body under the action of a set of nonconcurrent, coplanar forces.

Problems

1. Find the unknown forces **Q** and **R** at the right end of Fig. 101 by one moment and one resolution equation or by two resolution equations. Draw arrows inside the closed curve to show the direction of the forces acting *from* each member *upon* the joint. *Ans.* **R** = 160 lb compression; Q = 200 lb tension.

2. Using the value of Q found in Prob. 1 (or its components), find the magnitude of the forces **P** and **T** at the top. Draw the arrows.
   Ans. **T** = 241.70 lb tension; **P** = 166.92 lb tension; horizontal component of **P** = 118.03 lb.
3. At the left end, put an arrow in the horizontal member opposite the arrow **R** at the right end, and an arrow in the inclined member opposite the direction of **P** at the top. Assume that the horizontal hinge reaction is toward the right and the vertical hinge reaction is upward. (If the calculated value of **H** should come out negative, the meaning would be that the force is horizontal toward the left.) Find **H** and **V** and check. Find the resultant hinge reaction.
   Ans. **H** = 41.97 lb toward the right; **V** = 118.03 lb downward; resultant = 125.27 lb downward at 19°35′ to the right of the vertical.
4. Check **H** and **V** by moments about the top hinge, using the whole triangle as a free body. Check **T** by moments about the lower left corner.

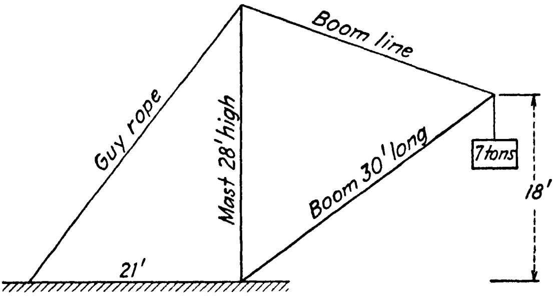

FIG. 102.

Figure 102 shows a simplified derrick with the boom placed opposite one of the guy ropes. In many practical problems the necessary data are in the form of dimensions rather than angles In this figure the dimensions have been chosen so that cosines of the angles with the horizontal and vertical are simple ratios from the diophantine[1] triangles 3,4,5 and 5,12,13, and the solution can be carried out with little labor by resolving on horizontal and vertical axes at each of the joints.

Problems

5. Find by the method of joints the forces in the four members of the derrick shown in Fig. 102 with a 7-ton load.
6. Find the magnitude and direction of the reaction of the ground at the junction of the mast and boom by considering this junction as a point in equilibrium.

---

[1] In honor of Diophantus, a fourth-century Greek mathematician of Alexandria, who devised the rule for constructing right-angled triangles: Take any two numbers m and n and one side the the triangle will be $m^2 - n^2$, a second will be $2mn$, and the hypotenuse will be $m^2 + n^2$.

7. Using the triangular frame as a free body, find the tension in the guy rope and components of the ground reaction.
8. Lay out a force triangle for the joint from which the load hangs. (By a proper choice of the order of the forces, the force triangle for this joint can be made similar to the triangle of the frame, and the values for the two unknown forces at the joint can be easily found by proportion rather than by scaling. This situation occurs so often in frameworks of two-force members that it is worth looking for, but in no case should the student superimpose the force diagram on the space diagram.)
9. Lay out the force diagram for the other two joints and scale the results. Transfer the direction of the ground reaction back to the scale drawing of the space diagram and show that it intersects the line of the 7-ton load at the same place as does the guy rope extended.
10. Superimpose the three-force diagrams (rearranging the order of the forces if necessary) so that each vector in every diagram lies along its corresponding vector in the others. The figure formed by these force diagrams is called a *Maxwell diagram* and is a very useful transformation of the problem. Bow's method for constructing and interpreting the diagram will be presented in the next chapter. One skilled in this art can take the Maxwell diagram and reproduce the structure to which it pertains in its proper proportions. If given a value for any dimension and any force, he can find all the others.

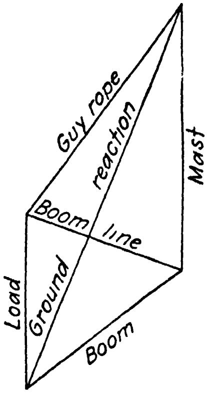

FIG. 103.

60. Jointed Frame with Three-force Members. In the problems just considered, all the loads on the frames were applied at the joints, the weights of the members were neglected, and no member was attached to three or more joints. The members were links, or two-force members. In practice it is often convenient and economical to build structures in which some or all of these conditions are not met, and such structures are usually built with joints which are reasonable approximations of smooth hinges. The members of such structures usually exert forces on the joints in *unknown directions* and the study of the equilibrium of such a joint is rarely useful. When such a structure is not statically indeterminate, a solution can be arrived at by considering each separate member as a free body. It is sometimes necessary and always advisable to consider the whole structure as a free body under the action of the outside forces. In the first of the two examples following, it is not possible to determine the external reactions without at least one equation taken from one of the bars as a free body. In the second example, the outside reactions R_1 and R_2 can be easily found first, and finding them is a good opening move.

Example 1

Figure 104 shows two bars AB and BC, which are hinge connected at B and attached to fixed hinges at A and C. To designate the forces, the bar AB is numbered 1 and the bar BC is numbered 2. The components of the reactions at A are H_1 and V_1. The components at C are H_2 and V_2. At B the components of the force which bar AB (No. 1) exerts on bar BC are H_{12} and V_{12} (Fig. 104,II) and the components of the force which bar BC exerts on bar AB are H_{21} and V_{21}. The component H_{21} is equal and opposite to H_{12}, and the component V_{21} is equal and opposite to V_{12}. When both bars are considered as a single free body, there are four unknowns, H_1, V_1, H_2, and V_2. When BC is considered as a free body there are four unknowns, H_{12}, V_{12}, H_2, and V_2. If both bars together are taken as a single body and moments are written about A, there are two unknowns, V_2 and H_2. If BC is taken as the free body and moments are written about B, the unknowns are again V_2 and H_2.

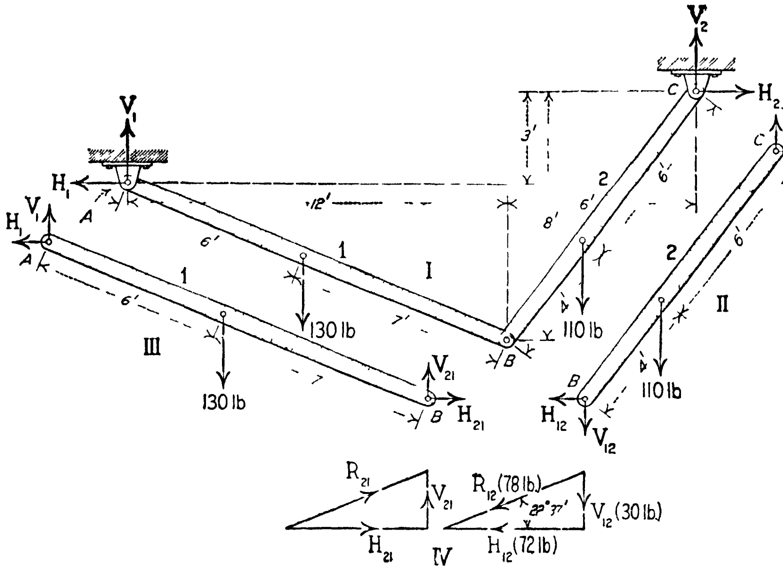

FIG 104

When moments are taken about A with the two bars together as one free body, the equation is

$$18V_2 - 3H_2 = 130 \times 6 \times \tfrac{12}{13} + 110 \times (12 + 0.6 \times 4) \qquad (1)$$

$$18V_2 - 3H_2 = 720 + 1584 = 2304 \qquad (2)$$

$$6V_2 - H_2 = 768 \qquad (3)$$

When moments are taken about B with BC as the free body,

$$6V_2 - 8H_2 = 110 \times 2.4 = 264 \qquad (4)$$

$$H_2 = 72 \text{ lb toward the right}$$

$$V_2 = 140 \text{ lb upward}$$

Resolutions with both members together as the free body give

$$H_1 = 72 \text{ lb toward the left}$$
$$V_1 = 100 \text{ lb upward}$$

If the bar BC (Fig. 104,II) is taken as the free body the resolutions give

$$H_{12} = 72 \text{ lb toward the left}$$
$$V_{12} = 30 \text{ lb downward}$$

The resultant force from AB to BC at B is 78 lb to the left and downward at an angle of 22°37′. The resultant reaction at A is 123.2 lb toward the left at 54°15′ above the horizontal. The resultant reaction at B is 157.4 lb toward the right at 62°47′ above the horizontal.

Example 2

Figure 105 shows an A frame, which rests on a smooth horizontal floor. The vertical reactions at the bottom are found as a problem of nonconcurrent forces with the entire frame as a rigid free body. With these reactions known, each member still has four unknown forces. In the horizontal tie DE the horizontal force H_{23} at the right end passes through the hinge at the left end, so that a moment equation about D contains only one unknown.

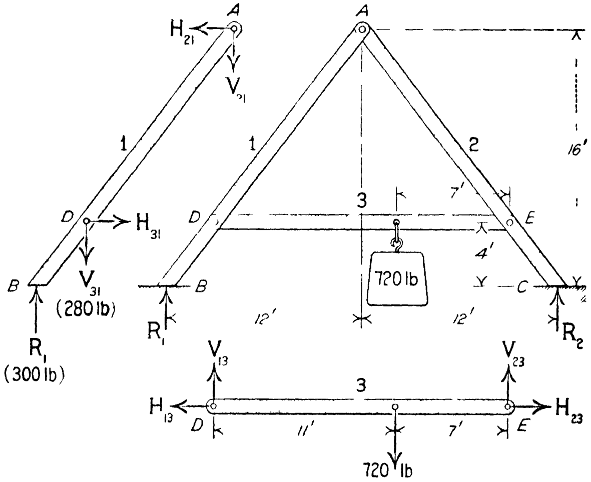

FIG. 105

$$V_{23} \times 18 = 720 \times 11$$
$$V_{23} = 440 \text{ lb acting upward on } DE$$

Moments about E give

$$V_{13} \times 18 = 720 \times 7$$
$$V_{13} = 280 \text{ lb}$$

To find the horizontal force H_{31}, with which the tie *DE* pulls on *ADB* at *D*, moments are taken about *A* with *ADB* as the free body.

$$H_{31} \times 12 + 280 \times 9 = 300 \times 12$$
$$H_{31} = 90 \text{ lb, pulling } ADB \text{ toward the right}$$

All the remaining forces may now be found by resolutions.

Problems

1. In Fig. 105, find the vertical force V_{21} at *A* by a vertical resolution. Then find H_{21} by moments about *D*. *Ans.* V_{21} = 20 lb down.
2. In Fig. 105, find H_{32} by moments about *A* with *AEC* as the free body and V_{32} known *Ans.* H_{32} = 90 lb, pulling *AEC* toward the left.

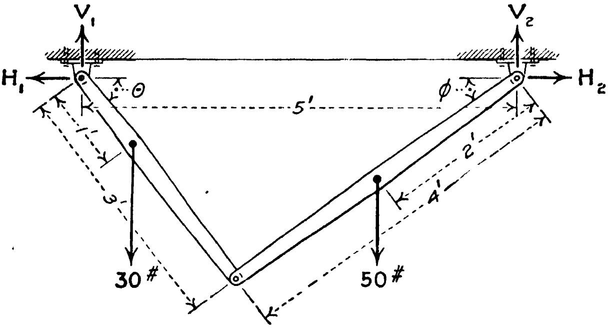

FIG. 106.

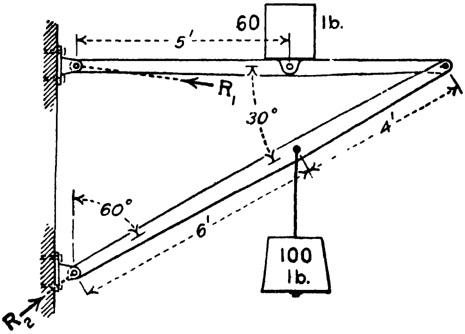

FIG. 107.

3. In Fig. 106, find the components of the reactions at the supports. Find the horizontal and vertical components, H_3 and V_3, of the force which the right bar exerts on the left bar at the lower joint.
   Ans. V_1 = 42.4 lb; H_1 = 16.8 lb toward the left; V_2 = 37.6 lb; V_3 = 12.4 lb downward.
4. In Fig. 107, find the direction and magnitude of the reaction at each of the supporting hinges, and R_3, the direction and magnitude of the force which the inclined member exerts on the horizontal member at the right hinge.

Ans. $H_1 = -H_2 = -H_3 = 163.92$ lb toward the left
$V_1 = 25.36$ lb $\quad V_2 = 134.64$ lb $\quad V_3 = 34.64$ lb
$R_1 = 165.9$ lb left 8°48′ up
$R_2 = 212.1$ lb right 39°24′ up
$R_3 = 167.5$ lb right 11°56′ up

5. In Fig. 108, find the direction and magnitude of the reaction at each hinge. Check.

6. Find the horizontal and vertical components of the reactions at the supports of Fig. 109, assuming weightless bars. (Horizontals are each 500 lb and one vertical is zero. The problem can be simplified by noting that the short member is a link.)

7. Solve Prob. 6 if the bars are uniform and weigh 20 lb per foot of length.

8. Solve Prob. 6 if the 866-lb load is replaced by a 1000-lb force downward to the right parallel to the shorter bar.

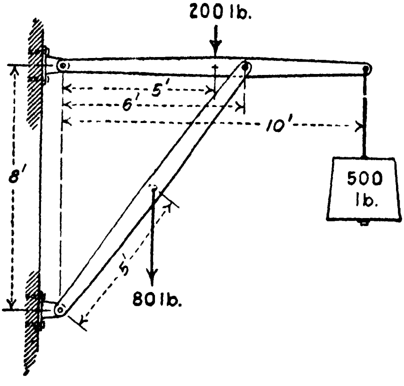

FIG. 108.

9. The figure-four framework in Fig. 110 is made of light bars hinged at the joints. The horizontal bar is 3 ft long and hinged at the middle to a point 2 ft above the bottom of the vertical bar and 4 ft below its top. The inclined bar is connected by a horizontal bar 2 ft long to a hinge on a wall 4 ft above the floor. With the load as shown, find the horizontal and vertical components of the forces acting on the vertical bar.

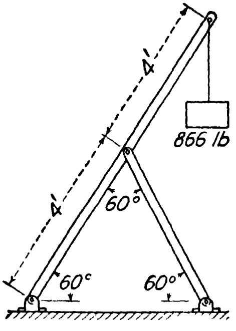

FIG. 109.

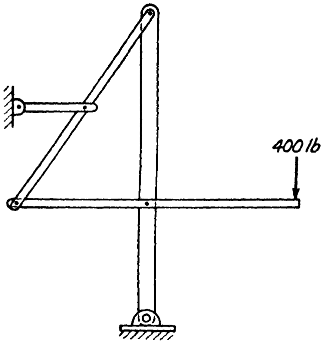

FIG. 110.

Ans. At the bottom: $V = 400$ lb upward
$H = 300$ lb to the right
2 ft from the bottom: $V = 800$ lb downward
$H = 450$ lb to the left
At the top: $V = 400$ lb upward
$H = 150$ lb to the right

10. In Prob. 9 let the horizontal bar from the wall be extended 1.5 ft and hinged to the vertical bar. Find the reaction at the bottom of the vertical bar.
Ans. The problem is indeterminate. By a suitable small movement of the pin at the bottom the forces can be made almost anything desired.

11. In Prob. 10 the reaction at the floor can be in any desired direction except along a line toward the hinge on the wall. Find the minimum value which it can have. *Ans.* 482 lb.

61. Trusses. A framework composed of relatively light members hinged together at the ends into connected triangular elements is called a *truss*. A truss with loads in its own plane *applied at the joints only* constitutes an extremely economical form of supporting structure for any system of loads or forces. Trusses have been devised in a wide variety of forms to meet special conditions, and the possible combinations are endless. A number of good types for meeting recurring conditions of loadings have been evolved during the last hundred years and the use of these types has tended to become standard practice. These trusses have been given names, usually after their original designer or place of origin, such as Whipple, Howe, Pratt, Warren, Fink, McCallum, King, Bollman, Pegram, Baltimore, Belgian, English, but sometimes the names are descriptive of some characteristic of appearance or function such as cantilever, belly-brace, fan, K, lenticular, panel, hammer-beam, bowstring, bed-stead, lattice, scissors, and the like. Texts on structural design usually discuss the advantages of the various types and indicate the conditions under which they are suitable or economical. This text will be confined to the basic methods of analysis applicable to any type of true truss, as these methods are both necessary and sufficient for all types

The basic element of a truss is the triangle, and a single triangle has three sides and three angles; in truss terminology, three members and three joints. For every additional triangle added onto the first, there will be added two sides and one angle, so that the number of members of a statically determinate truss in a plane will be three less than twice the number of joints. This rule can be used to determine whether the truss is statically determinate or not. The ideal truss is composed of two-force members only, and the basic truss analysis assumes the following three conditions:

1. The members are weightless.
2. The members are joined together only at their ends and these joints are all perfectly frictionless hinges.
3. There are no loads on the structure except at the joints.

It is obvious that no real truss meets these conditions exactly, but for the great majority of real trusses (1) the weights of the members are negligible compared with the loads which the structure carries; (2) the flexibility of the members is sufficient to compensate in a large part for the rigidity of the joints; and (3) the loads are very nearly centered on the intersections of the center lines of the members. In cases where the assumptions of the ideal truss are not justifiable or where a more refined analysis is desired, it is the usual practice to make a basic analysis as though the truss were ideal and to follow this with a study of the "secondary stresses" set up by the distributed weights of the members and the rigidity of the joints, a study involving the elasticity of the members. The basic analysis then includes the weights of the members but treats them as divided proportionately between the two ends of each member so that they fall on the joints of the structure.

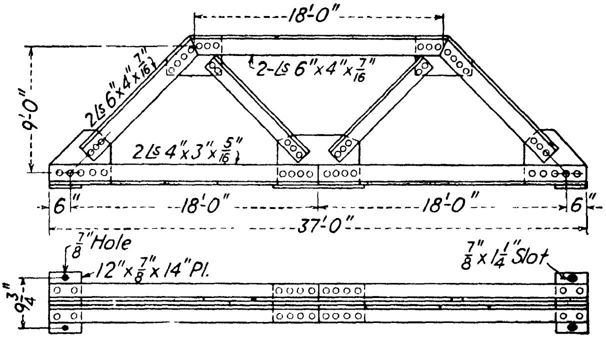

FIG. 111.

If this were not done the analysis would be complicated by the necessity of taking the members rather than the joints as free bodies. In the example below the weights of the members are neglected and the pins at the joints are considered weightless free bodies. Figure 111 shows a riveted steel truss as it would actually be constructed. This truss would weigh about 1 ton and would support safely over 75 tons divided equally among the three joints.

In Fig. 112 a beginning may be made at D. The known force is 1200 pounds downward. The unknowns are T_{cd} in the member CD and Q_{gd} in the member GD. Resolving vertically,

$$Q_{gd} \sin CDG = 1200$$
$$Q_{gd} = 1200 \times {}^{17}\!/_{8} = 2550 \text{ lb compression}$$

Resolving horizontally,

$$T_{cd} = Q_{gd} \cos CDG = 2550 \times {}^{15}\!/_{17} = 2250 \text{ lb tension}$$

At joint C the horizontal tension of 2250 pounds is now known, but there are still three unknowns. At joint G the only unknowns are the vertical force V_{cg} and the horizontal force P_{fg}. Resolving vertically,

$$V_{cg} = 400 + 2550 \cos DGC = 400 + 1200 = 1600 \text{ lb tension}$$

Since it is known that the vertical component of the compression in GD is 1200 pounds, it is not necessary to multiply 2550 by the cosine of the angle DGC. Resolving horizontally,

$$P_{fg} = 2250 \text{ lb compression}$$

There are now only two unknowns at C. A vertical resolution gives

$$Q_{fc} \times 0.8 = 1600 \qquad Q_{fc} = 2000 \text{ lb compression}$$
$$T_{bc} = 2250 + 0.6 \times 2000 = 3450 \text{ lb tension}$$

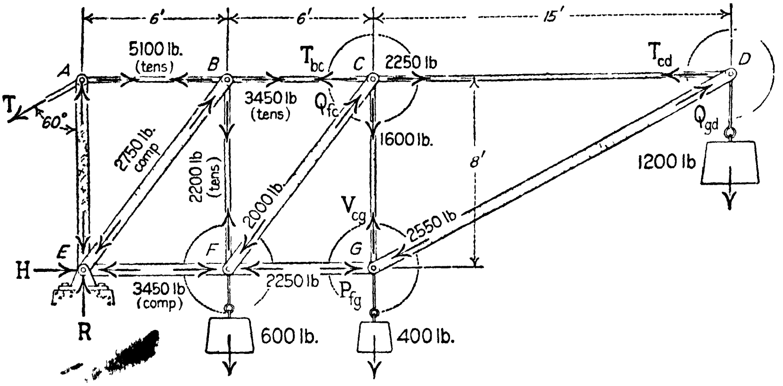

FIG. 112.

Problems

1. Find the remaining internal forces of Fig. 112, one joint at a time. Find the external forces H, R, and T.
2. Find the external forces of Fig. 112 as a problem of nonconcurrent forces.

Figure 113 shows a Warren truss with all members of the same length. The symbol at the left end indicates rollers or a device to ensure that this reaction is vertical only. The symbol at the right indicates a hinge, at which there are both horizontal and vertical components if the loads have any horizontal components. Every truss should be hinged at one end and free to move at the other without great friction. If both ends are hinged or fixed, large thermal stresses can develop. If neither end is fixed or hinged and friction is depended on to resist lateral movement, it may happen that the truss moves at one end only upon expansion and at the other on contraction. The collapse of a theater roof on a houseful of children in Washington, D.C., was attributed to this cause.

In Fig. 113 it is not possible to start at any joint until one of the reactions has been found. Moments about the hinge give $R_1 = 7500$ lb, and then a start can be made at the lower left joint, but before starting the solution by the method of joints, it is best to get R_2 and H, and check. Then if the final joint checks out, the whole problem is checked.

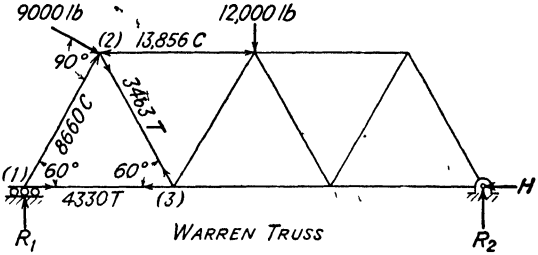

FIG. 113.

Problem

3. Find R_2 and H. *Ans.* $R_2 = 9000$ lb; $H = 7794$ lb.
(Since, in this case, the 9000-lb load is perpendicular to the end post, it is convenient to use the whole force in taking moments about R_1 and to use its vertical and horizontal components in taking moments about R_2.)

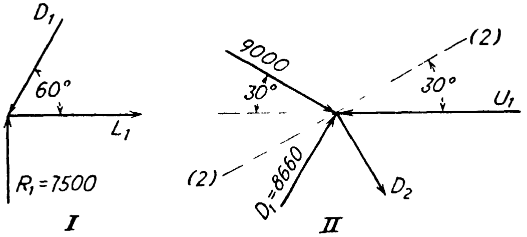

FIG. 114

Figure 114,I shows the lower left joint isolated as a free body. Resolving vertically,

$$D_1 \cos 30° = 7500 \text{ lb} \qquad D_1 = 8660 \text{ lb tension}$$

Resolving horizontally,

$$L_1 = D_1 \cos 60° \qquad L_1 = 4330 \text{ lb compression}$$

Figure 114,II shows the upper left joint isolated as a free body. The value 8660 for D_1 is brought forward from the joint previously solved. Directions of U_1 and D_2 are assumed, U_1 in compression, D_2 in tension. (If a value comes out as a negative it will mean simply that the opposite is true.) A vertical resolution is done first, as it will eliminate U_1 and yield an equation in one unknown. Resolving vertically,

$$9000 \cos 60° + D_2 \cos 30° - 8660 \cos 30° = 0 \qquad D_2 = 3463 \text{ lb tension}$$

Resolving on axis (2),

$$9000 \cos 60° + 8660 \cos 30° = U_1 \cos 30°$$
$$9000 \tan 30° + 8660 = U_1 \qquad U_1 = 13{,}856 \text{ lb compression}$$

The second equation might just as well have been a horizontal resolution. Axis 2 perpendicular to D_2 eliminated a value already found and simply reduced the labor by one term in the equation. It is obvious that, with a large clear sketch for Fig. 113, Figs. 114,I and II are not really necessary, as the work can be done directly from the complete sketch. It is probably better for a beginner to make a separate sketch for each joint until he has acquired considerable skill. If the work is done directly on the complete sketch it is not advisable to run all over the sketch putting on arrowheads in assumed directions; it is better to complete each joint, then carry the arrowheads forward to the opposite ends of the members connected to it, to indicate the forces already found. In this way a member without an arrow indicates an unknown, and it is obvious at a glance that a joint can or cannot be solved.

It is necessary to have some systematic way of designating the various members and their forces, and many good systems have been devised. The one used above assigns to each member of the lower chord the letter L with subscript 1, 2, 3 reading from left to right. The diagonals are D_1, D_2, D_3, etc. These diagonals are usually called *web members*, the outer ones being the *end posts*. The upper chord members are U_1 and U_2. A very common notation is to designate the *upper joints* as U_1, U_2, etc., and the lower joints as L_1, L_2, etc. The members are then referred to as U_1L_1, U_1L_2, etc., and are so labeled in the shop as a guide to erection in the field or assembly in the shop.

Problems

4. Find the forces in all members of Fig. 113 by the method of joints. Figure 113 shows how the sketch should appear after the first two joints have been solved and it is obvious that joint (3) has but two unknowns while the center top joint has three. The next to last joint will have only one unknown but will check

out if the preceding work has been done correctly. The last joint will have no unknowns, but it should be checked by two resolutions as an over-all check on the whole problem.

5. Figure 115 shows a truss 16 ft high with six panels of 12 ft each. With a 46-ton load distributed as shown, find the reactions and check.

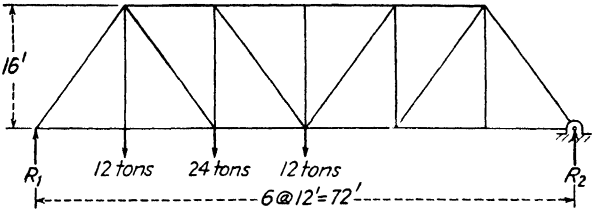

FIG. 115.

6. Solve for forces in all members by the method of joints. (Start at either end. Resist the temptation to move forward before a joint has been completely solved. The problem is set up to give answers in round numbers.)
7. Figure 115 is a Pratt truss. If the diagonal members (except the end posts) are reversed, the truss becomes a Howe truss. Make this change and solve again.

62. Method of Sections. In the preceding article, each joint was considered as a free body in equilibrium under concurrent forces. It is also possible to divide a jointed frame into two portions by an imaginary surface and to treat either portion as a free body in equilibrium under nonconcurrent forces. Suppose the truss of Fig. 116 is divided by a surface which passes through the members BC, FC, and FG, and the portion to the right of this surface is taken as the free body. This portion CDG, Fig. 116, is subjected to the loads of 1200 pounds and 400 pounds, and the forces from the left portion which act through the members BC, FC, and FG. All these are external forces with

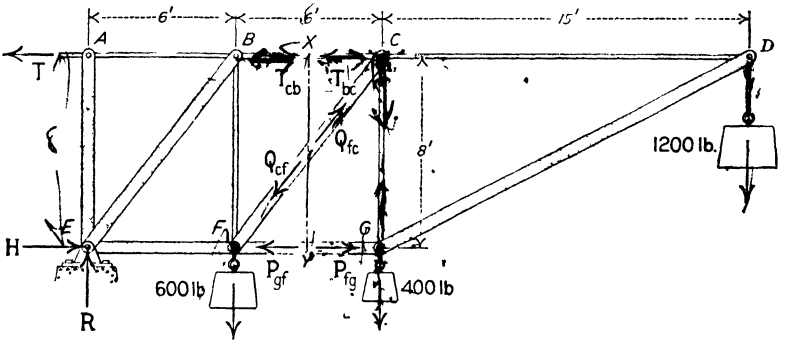

FIG 116.

respect to the free body under consideration. Since there are only three unknowns, the problem can be solved.

Moments about F eliminate the forces in FC and FG.

$$\begin{aligned} 1200 \times 21 &= 25{,}200 \\ 400 \times 6 &= 2{,}400 \\ \hline 8T_{bc} &= 27{,}600 \\ T_{bc} &= 3450 \text{ lb tension} \end{aligned}$$

By moments about C,

$$\begin{aligned} 8P_{fg} &= 1200 \times 15 \\ P_{fg} &= 2250 \text{ lb compression} \end{aligned}$$

By vertical resolutions,

$$\begin{aligned} 0.8Q_{fc} &= 1600 \\ Q_{fc} &= 2000 \text{ lb compression} \end{aligned}$$

The portion of the truss to the left of the surface XY of Fig. 116 may be taken as the free body. The external forces are the load of 600 pounds, the reactions **R**, **H**, and **T**, and the forces $\mathbf{T}_{cb}$, $\mathbf{Q}_{cf}$, and $\mathbf{P}_{gf}$, acting on the left portion from the right portion. (To indicate that the forces under consideration act from the right portion upon the left portion, the subscripts have been reversed. The force $\mathbf{T}_{bc}$ is opposite to the force $\mathbf{T}_{cb}$, etc.) The external reactions **R**, **H**, and **T** are first calculated from the entire truss as a free body. The results are $\mathbf{H} = \mathbf{T} = 5100$ pounds; $\mathbf{R} = 2200$ pounds. By vertical resolution,

$$\begin{aligned} 0.8Q_{cf} + 600 &= 2200 \\ Q_{cf} &= 2000 \text{ lb} \end{aligned}$$

The vertical component is downward, hence the stress in the member FC is compression.

By moments about F,

$$\begin{aligned} 8T_{cb} &= 8 \times 5100 - 6 \times 2200 \\ T_{cb} &= 3450 \text{ lb} \end{aligned}$$

Since the moment is clockwise about F, the force T_{cb} acts toward the right and the stress in CB is tension.

If a diagonal section is drawn on Fig. 116 cutting the members CD, CG, and FG, the bar GD is separated as a free body. A horizontal resolution shows that the tension in CD is equal to the compression in

FG. A vertical resolution shows that the tension in *CG* is 1600 pounds.

The method of sections is of extreme importance in the practical business of designing trusses. The problem usually presented is that of making every member (and joint) of sufficient strength to sustain the greatest force which can fall upon it under any possible combination of loads, so that each member is designed for the particular combination of loads which causes the greatest force in that member. Without the method of sections or its equivalent, by which the force in any member may be calculated directly for any combination of loads, a great deal of time would be wasted in calculating useless forces in other members. A clear understanding of the method and skill in its application also enable the designer to determine the effect on any member of a load placed at any joint of the truss, and so to draw what is called the *influence line* for that member.

From the example it should be clear that a section which cuts three two-force members of a truss (all the outside forces being known) makes either portion of the truss a solvable problem, the three unknowns being the *magnitudes of the forces in the cut members.* If four members are cut, the problem is not solvable unless the force in one of the cut members is known and is used in the solution. If a three-force member is cut, in a framework containing that kind of member, the exact position of the section on the member is important, and at that section there are *three unknowns*, the force in the direction of the member, the force (shear at the section) perpendicular to the member, and a couple (the moment at the section).

Problems

1. Make a freehand sketch of the truss of Fig. 116. Pass a section through *AB*, *EB*, and *EF*. Find the stress in each of these members, using the portion to the right of the section as a free body.
2. Solve Prob. 1, using the post *EA* as the free body.
3. On your sketch of the truss of Fig. 116 draw a diagonal section cutting the members *BC*, *BF*, and *EF*. Find the stress in *BF*. Show that the tension in *BC* is equal to the compression in *EF*. Find the compression in *EF*.
4. On your sketch of the truss of Fig. 116 draw a section cutting the members *BC*, *FC*, *CG*, and *GD*. Can you find the stress in each of these members, using the portion to the right of the section as the free body? Why?
5. In Fig. 116, the force **T** is horizontal, while it makes an angle of 30° with the horizontal in Fig. 112. What effect does this difference in angle make on the external forces? On the internal stresses?
6. Find the reactions at the supports of Fig. 117. Resolve the 4200-lb force into its horizontal and vertical components to take moments about *D*. Check by vertical resolutions. Check again by moments about *A*.

7. Make a sketch of Fig. 117, showing the reactions and loads. Draw a section cutting AB, FB, and FE. Find the stress in each of these members, using the portion to the right of the section as the free body. Check, using the portion to the left of the section as the free body.

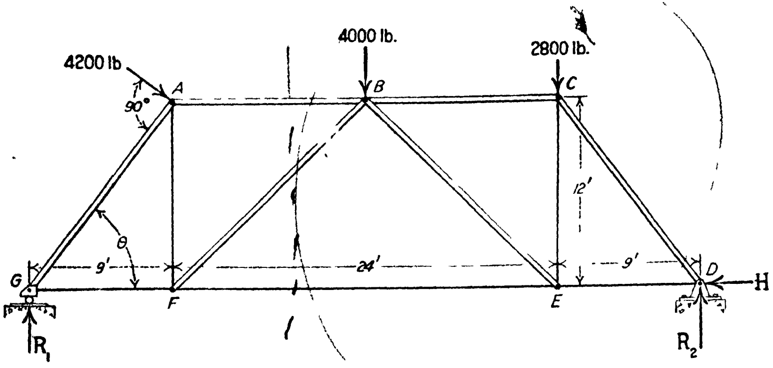

FIG. 117.

8. Begin at G of Fig. 117 and find all stresses, one joint at a time. Check stresses in BC, EC, and ED by sections.

63. Miscellaneous Problems

1. In a triangular frame ABC, the member AB is horizontal and is 14 ft long. The members AC and BC are each 9 ft long. The joint C is vertically above the member AB. The frame is supported by a vertical reaction at the left end A and is attached to a fixed hinge at B. Find the stress in each member when a load of 160 lb is applied at C.
2. In the frame of Prob. 1, the load of 160 lb makes an angle of 30° to the right of the vertical downward. Find the reaction and the stress in each member. Check. Show that the stress in AB is always seven-ninths of the stress in AC.
   Ans. Stress in BC = 161.6 lb compression.
3. In a quadrilateral frame the member AB, 20 ft long, is horizontal, is supported at the left end A, and is attached to a fixed hinge at B. The member CD, 12 ft long, is 12 ft above AB with the left end C 4 ft to the right of the vertical line through A. The joint A is connected by bars to C and to D, and the joint B is connected to D. A load of 400 lb is applied at C and a load of 800 lb at D. Find the stress in AD before finding any other internal stress. Find the force in AB and the force in BC without using any other internal stress. Begin at B and solve for all stresses, one joint at a time.
4. In the frame of Prob. 3, the 800-lb load is vertical and the 400-lb load at C is perpendicular to AC. Find the stress in each member and check.
5. In the frame of Prob. 3, there is a vertical load of 800 lb at D and a vertical load W at C. What is the value of W if there is no stress in AD? If there is 120 lb of compression in AD?
6. A rectangular frame, hinged at the corners, is hung from one corner and carries a load W at the opposite corner. A bar connects the remaining two corners. Find the compression in this bar in terms of W.

7. A 20-lb rectangular block and an 18-lb rectangular block rest on a horizontal plane with a 22-lb cylinder between them. The coefficient of friction between the plane and any one of the three bodies is 0.5, and the coefficient of friction between the cylinder and the vertical faces of the blocks in contact with it is 0.2. If the axis of the cylinder is horizontal and perpendicular to the direction of the push, what horizontal push on the 20-lb block will move the three bodies uniformly along the plane? *Ans.* 21 lb.

8. For what range of weights of the cylinder in Prob. 7 (other values unchanged) would the answer come out half the sum of the weights of the three bodies?

9. A rectangular block is placed on a 30° inclined plane and a cylinder weighing half as much as the block is laid behind the block. If the block rolls down the plane uniformly, pushing the cylinder before it, what is the coefficient of friction if it is assumed to be the same between the cylinder and the block as between the block and the plane? *Ans.* $\mu = 0.67$.

10. Generalize Prob. 9 by assigning the block a weight B, the cylinder a weight C, and making the angle of inclination θ.

11. A derrick boom is 40 ft long and makes an angle of 30° with the horizontal (see Fig. 125, p. 120). The mast is 30 ft high. One guy rope in the vertical plane of the mast and boom makes an angle of 35° with the horizontal. Find the tension in the guy rope when the load is 1200 lb. Solve as a problem of nonconcurrent forces with the mast, boom, and the rope which connects them as the free body. *Ans.* $T = 1691.6$ lb.

12. In Prob. 11, find the horizontal reaction at the bottom of the mast by moments about the top of the mast. Find the vertical reaction at the bottom of the mast by moments about the lower end of the guy rope. Check by a vertical resolution.

13. In the derrick of Prob. 11, find the compression in the boom and the tension in the rope which supports it as a problem of concurrent forces.

Ans. 1600 lb; 1442.3 lb.

14. Using one answer of Prob. 13, find the tension in the guy rope and the compression in the mast at the top as a problem of concurrent forces.

Ans. Compression in mast = 1370.3 lb.

15. Using one answer of Prob. 13 and one answer of Prob. 14, find the horizontal and vertical components of the reaction at the bottom of the mast as a problem of concurrent forces. Compare with the answers of Prob. 12.

16. In Fig. 118, the two diagonal members are connected by a horizontal rope at the top. The horizontal bar is pinned at one end and rests on a smooth pin at the other. Neglect the weight of the members and analyze the structure with the 800-lb load shown. (*a*) Show each part separately with the forces acting on it. (*b*) Find the shear and the moment at the middle of the horizontal bar. (*c*) Find the shear, moment, and axial force at the middle of one of the inclined members.

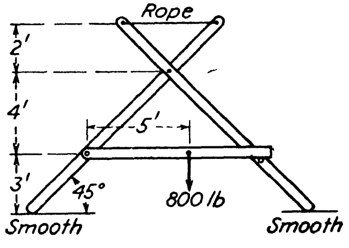

FIG. 118.

17. Remove the 800-lb load from the bar of Prob. 16 and hang it from the pin connecting the two inclined members, then find the tension in the rope and the forces acting on each of the two inclined members. Finally take the pin from which the load is hung and check it for equilibrium.

18. Solve Prob. 17 with the load at the middle of the rope instead of at the pin. (Note that one of the legs will slide on the floor and the angle 45° will change.)

19. In Fig. 118, remove the rope and the 800-lb load. Pin the horizontal bar at both ends and lay a smooth 707-lb cylinder between the bars at the top, letting the size of the cylinder be such that the points of contact are 2 ft from the upper pin. Show each part separately as a free body and find the components of the forces acting at the pins.

20. A light horizontal bar AB 13 ft long is hinged at its right end A and has a small smooth pulley at its left end B. A light bar CE 14 ft long is hinged to an outside support at C 6 ft below A and hinged to AB at D 8 ft from A. A cord attached at E runs over the smooth pulley at B and supports a load of 300 lb. Find the components of the reactions at A and C and of the intermediate force at D.

21. In Prob. 20, find the shear, moment, and axial force in each of the two bars just to the left of D and also just to the right of D.

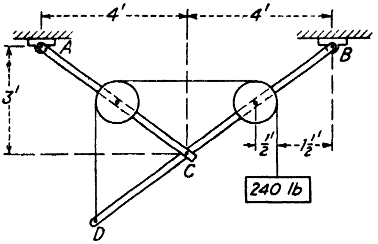

FIG. 119.

22. Figure 119 shows two weightless bars hinged at A and B and pinned together at C. The pulleys are frictionless and of negligible weight. They are 1 ft in diameter and have their center midway between A and C and B and C. Find the external forces on each of the two bars. Show graphically or otherwise that the external forces at A and B intersect on the line of the vertical cord supporting the 240-lb body. Find the forces at all pins.

23. A bar AB is hinged to an outside support at A, and B is 4 ft below and 3 ft to the right of A. Bar BC is hinged to AB at B, with C 5 ft to the right of B. Bar BC weighs 40 lb and is held at C by a cord making an angle of 30° with the horizontal. Find the weight of AB. *Ans.* 91.66 lb.

24. A light truck weighing 3000 lb with four-wheel drive on a horizontal surface where the coefficient of friction is 0.6, is attached by a horizontal cable to a 3-ton body resting on the same surface. It fails to start the body because the coefficient of starting friction between the body and the surface is 0.4. At what angle with the horizontal must the cable be placed to enable the truck to drag the body? *Ans.* Over 22°37′.

25. Two bodies of equal weight, placed on a 20° inclined plane where the coefficient of friction is 0.5, are connected by a cord. It is required that the two bodies move uniformly down the plane when rollers are placed under the front body. What should be the angle of the cord?

26. A 40-lb mass slides on a smooth straight rod which makes an angle of 25° with the horizontal. It is attached by means of a weightless cord to a mass of 20 lb which slides on a second smooth rod. This rod makes an angle of 35° with the horizontal. The two rods are in the same vertical plane and on opposite sides of the vertical. Find the direction of the cord, the tension which it exerts, and the normal reaction of each rod.

CHAPTER 7

METHOD OF WORK

70. Energy Methods. In the science of dynamics, equations in which the total energy in a system of bodies at one time or in one condition, plus the energy put into the system and minus the energy taken out of the system, is equated to the total energy remaining in the system at a later time or in a changed condition or configuration constitute one of the most powerful and general methods of mathematical analysis. There are in statics many practical problems in which the changes of the velocities of masses are zero or so small that the energy involved in such changes can be neglected and in which the magnitudes of the forces suffer little or no change during the motions. In such cases some unknowns (usually the ones of practical interest) can often be found by energy equations without the necessity of solving for many other unknowns involved in the problem.

Example

The 400-lb cylinder of Fig. 145 is held on the 30° rough inclined plane by a rope attached to the cylinder and coming off at the top tangent to the cylinder and parallel to the plane. The rope passes over a fixed pulley, thence vertically downward around a moving pulley and up to a point of attachment. Find the weight of W required to hold the cylinder in place on the plane.

Suppose the center of the cylinder to move upward parallel to the plane a distance z and the weight W to descend a distance y. The work done by gravity on the weight W is Wy and the work done against gravity on the 400-lb cylinder is $400z \sin 30°$.

Equating,

$$Wy = 400z \sin 30° \qquad (1)$$

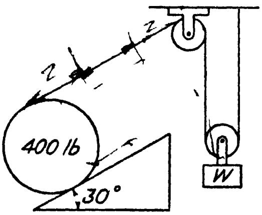

FIG. 145.

When the cylinder rolls a distance z up the plane, it turns on its axis and a length of rope z is unwound from the cylinder. A length $2z$ passes over the top pulley and a length z moves from one side to the other of the lower pulley, so this pulley moves down a distance z and

$$y = z \qquad (2)$$

Substituting in Eq. (1),

$$W = 400 \sin 30° = 200 \text{ lb}$$

This illustration neglects work dissipated by the friction of the pulleys. $W = 200$ pounds is sufficient to hold the cylinder if there is no friction. To start the body up the plane, W will have to be more than 200 pounds, or to permit it to start down the plane, W will have to be nearly the same amount less than 200 pounds. At the point of contact the plane exerts a force of friction on the cylinder, but there is *no slipping* and no work is done by this force. It was properly omitted from the equation. If the pulley friction must be put into the problem, it is necessary to determine the friction forces between the pulleys and their axles, to multiply each friction force by the distance of slipping at its point of application, and to add this sum to the right-hand member of Eq. (1). An alternate method is to estimate the *efficiency* of the machine and to multiply the left-hand member of Eq. (1) by the efficiency, the efficiency being defined as the ratio of the energy output of the machine to the energy input.

71. Virtual Work. In the illustration above, y and z can be made any desired distances within the limits of motion of the apparatus since (if the weight of the rope is neglected) the force system is the same for any position of the bodies. But if the weight of the rope is to be included as an element of the problem or the ropes have directions other than vertical and parallel to the plane, the forces change with the position and the distances used should be infinitesimal. Equation (1) then becomes $W\,dy = dz \sin 30°$, where y is the distance of W above some arbitrary level line and z is the distance of the center of the cylinder from some arbitrary point on its path. When the sum of the work done (positive and negative) by all the forces acting on a body (or system of bodies) during an infinitesimal motion of the body (or a conformal pattern of infinitesimal motions of the system of bodies) is equated to zero, the equation is called an *equation of virtual work*, and the method of solving statics problems by such equations is called the *method of virtual work*.[1]

The definitions of work and of equilibrium combine to give a proof of the validity of the principle of virtual work. Work is the (scalar) product of force and displacement. A body in equilibrium is acted upon by a system of forces whose vector sum is zero. Therefore, when such a body is given a displacement (assuming equilibrium throughout the displacement) the net work done by the displacing force is the product of the displacement and the resultant of the other forces.

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[1] The term *virtual work* survives in the literature of mechanics with the meaning of *virtual velocity*, as used by the early writers on mechanics. The nearest modern dictionary definition of the adjective *virtual* as used in this term is "being in essence or effect but not in fact."

is resultant being zero, the product is also zero. Similarly, the lar product of the couples and the angular displacement must be ro when the system of couples acting on the body is in equilibrium.

many problems of statics, especially those in which friction can be glected, the method of virtual work gives the most elegant solutions d the method can easily be extended in many cases to make it yield a ar, definite determination of the class of equilibrium.

Example 1

Figure 146 shows a bar of length l and mass W, with its center of mass at a tance a from the lower end. The bar rests on a smooth horizontal floor and ns against a smooth vertical wall. The bar eld from slipping by a horizontal force at bottom. This horizontal force will be computed by the method of work.

When the lower end of the bar is moved a distance dx toward the right, the center of mass is moved upward a distance dy. As the reactions at the wall and the floor are normal to the displacement, their work is zero. This work equation is

$$P\,dx = W\,dy \tag{1}$$

The relation between dx and dy is found by differentiating an equation between x and y. This relation is conveniently found by means of a third variable. If θ is the angle between the bar and the horizontal, and y is the distance of the center of mass above the floor,

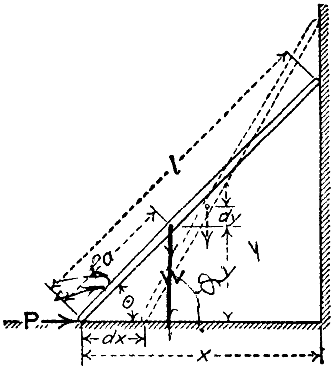


FIG. 146.

$$y = a \sin\theta \tag{2}$$
$$dy = a \cos\theta\, d\theta \tag{3}$$

If x is the distance of the lower end of the bar from the vertical wall and is positive toward the right,

$$x = -l\cos\theta \tag{4}$$
$$dx = l \sin\theta\, d\theta \tag{5}$$

Substitution in Eq. (3) gives

$$Pl \sin\theta\, d\theta = Wa \cos\theta\, d\theta \tag{6}$$
$$\frac{W}{P} = \frac{l\tan\theta}{a} \tag{7}$$

Problems

1. A ladder 20 ft long, weighing 36 lb, with its center of gravity 8 ft from the lower end, rests on a smooth horizontal floor and leans against a smooth vertical wall. It is held from slipping by a horizontal force of 16 lb applied 2 ft from the bottom. Find the position of equilibrium by the method of virtual work. Check by moments and resolutions. *Ans.* 45° with the horizontal.
2. The ladder of Prob. 1 rests on a smooth horizontal floor and leans against a smooth wall which makes an angle of 15° with the vertical, away from the

ladder. It is held from slipping by a horizontal push of 8 lb at the bottom. Find the position of equilibrium.

Ans. $\tan\theta = 1.5321$; $\theta = 56°52'$ with the horizontal.

3. Find the tension in the rope of Prob. 16, Art. 63, by the method of virtual work (Let the length of the cord be x and the height of the point of attachment of the 800-lb load above the floor be y. Change the 45° to θ and express x and y in terms of θ. The virtual work equation is

$$-P\,dx = 800dy)$$

Ans. $P = 600$ lb

4. Does the force in the rope vary with the position of the load on the horizontal bar? What would be the force in the rope if the rope were shortened until the bars made angles of 50° with the floor?

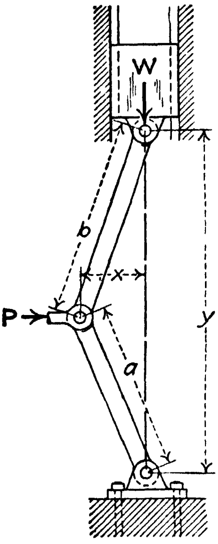


Fig. 147.

Example 2

Find the ratio of W to P in the toggle joint shown in Fig. 147.

$$-P\,dx = W\,dy$$

$$y = \sqrt{a^2 - x^2} + \sqrt{b^2 - x^2}$$

$$dy = \frac{-x\,dx}{\sqrt{a^2 - x^2}} + \frac{-x\,dx}{\sqrt{b^2 - x^2}}$$

$$\frac{W}{P} = \frac{\sqrt{(a^2 - x^2)(b^2 - x^2)}}{xy}$$

Example 3

Figure 148 shows the crank, connecting rod, and crosshead of an engine. Find the relation of the total piston pressure P to the component of the force perpendicular to the crank. Give the results in terms of the lengths of the crank and the connecting rod and the position of the crank

$$x = a\cos\theta + b\cos\phi = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}$$

$$-P\,dx = Ra\,d\theta$$

$$\frac{R}{P} = \sin\theta + \frac{a\sin\theta\cos\theta}{\sqrt{b^2 - a^2\sin^2\theta}} = \sin\theta + \cos\theta\tan\phi$$

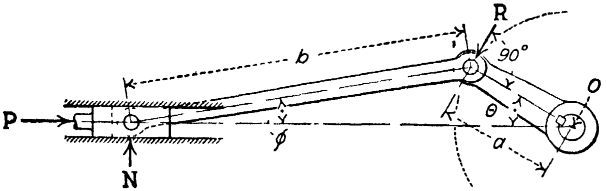


Fig. 148.

Problems

5. In Fig. 148, find the normal reaction **N** at the crosshead by resolutions. Then find the moment about the axis of the shaft at O by a moment equation, regarding the connecting rod and crank as a single body.

Compare the result with the answer of the example.

6. In Fig. 147, let $x = 1$ ft and $a = b = 8$ ft. Find P when $W = 1200$ lb.

7. A bar AB (Fig. 149) of length c is hinged at A and supported by a rope at B. The rope runs over a smooth pulley C at the same level as A, at a distance b therefrom, and carries a load of P lb on the free end. The bar weighs W lb and its center of mass is at a distance a from the hinge. Find the ratio of W to P at any position.

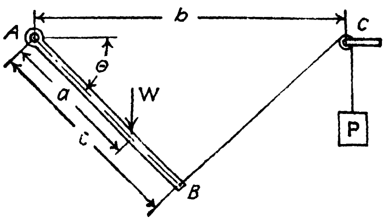


FIG. 149.

Let $BC = s$, and let the vertical distance of the center of mass below the hinge be represented by y, which is taken as positive downward.

$$W\,dy = P\,ds$$

$$\frac{W}{P} = \frac{bc \sin\theta}{a\cos\theta\sqrt{b^2 + c^2 - 2bc\cos\theta}}$$

8. In Prob. 7, let $W = 12$ lb, $P = 10$ lb, $a = 4$ ft, $b = 8$ ft, and $c = 6$ ft. Find the equation for the angle θ. *Ans.* $0.96\cos^3\theta - 2\cos^2\theta + 1 = 0$.

9. Solve Prob. 8 for $\cos\theta$ by the method of trial and error.

$\cos\theta$	$0.96\cos^3\theta$	$-2\cos^2\theta$	$f(\theta)$
1	0.96	−2	−0.04
0	0	+1	+1

The function of θ changes sign between $\cos\theta = 0$ and $\cos\theta = 1$. There must be one real root or three real roots between these values. One root is evidently much closer to unity than to zero. Beginning with $\cos\theta = 0.9$,

0.9	0.69984	−1.62	+0.07984
0.95	0.82308	−1.805	+0.01808
0.96	0.84934	−1.84320	+0.00614
0.97	0.87616	−1.88180	−0.00564
0.965	0.862687	−1.862450	+0.000237
0.966	0.865372	−1.866312	−0.000940

An interpolation of the last two results gives $\cos\theta = 0.9652$, $\theta = 15°10'$.

The remaining roots may be found by a similar process. It is better, however, to divide the cubic equation by $\cos\theta - 0.9652$. The quotient is an equation of the second degree, which may be solved by the ordinary methods for quadratics. One of the roots of this quadratic is a solution of the problem of mechanics. The other root is a solution of the mathematical equation but is mechanically impossible.

Draw the space diagram for the possible angle.

72. Classes of Equilibrium. In mechanics as applied to practical problems it is often necessary, in addition to determining that the conditions of equilibrium are satisfied, to determine whether the equilibrium secured is *stable, unstable,* or *neutral.*

Figure 150 shows three examples of *stable* equilibrium. When a body in stable equilibrium is displaced slightly from its position of equilibrium by an additional force, it will return to that position of equilibrium when the additional force ceases to act. In Fig. 150,I the body, which is hung on a flexible cord, is in stable equilibrium in position A. The cord is vertical, and the center of mass is at its lowest point, directly under the support. If an additional force is applied, which deflects the body to position B, it will return to position A when the additional force is removed.

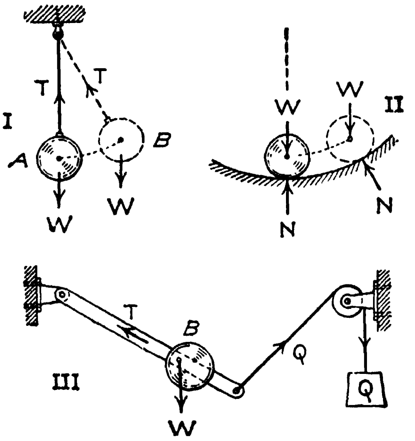


Fig. 150.

When a body in stable equilibrium is displaced, the additional forces do positive work, and the work of the original forces is negative. When the *additional* forces cease to act, the *original* forces do positive work to bring the body to its original position.

In Fig. 150,I when the body is moved from position A to position B, no work is done by the tension in the cord, since the displacement at any instant is perpendicular to the direction of the force. Negative work is done by the weight, since the upward component of the displacement is opposite to the direction of the force of gravity. The total work of the original forces is negative when the body is displaced, and the total work of these forces is positive when the body returns to its original position.

Figure 150,II shows a body on a smooth curved surface, which is concave upward. The force situation is the same as that in Fig. 150,I. Figure 150,III shows a body suspended from a smooth hinge and attached to a cord which runs over a smooth pulley and supports a mass Q. The reaction at the hinge does no work when the right end of the bar is moved toward the pulley, but the force Q does positive work and the force W does negative work. It may be shown that in this position the negative work is greater than the positive work, and work must be supplied by the additional force to effect the motion. The equilibrium is stable. The same may be said of a motion of the bar in the opposite direction except that the negative work of Q is greater than the positive work of W.

Figure 151,I, II, and III shows three cases of *unstable* equilibrium, which are the exact opposites of those shown in Figure 150. In many cases, such as III of these figures, an analysis of the *position of equilibrium* yields two answers, one for the stable and one for the unstable position, without any indication as to which is which.

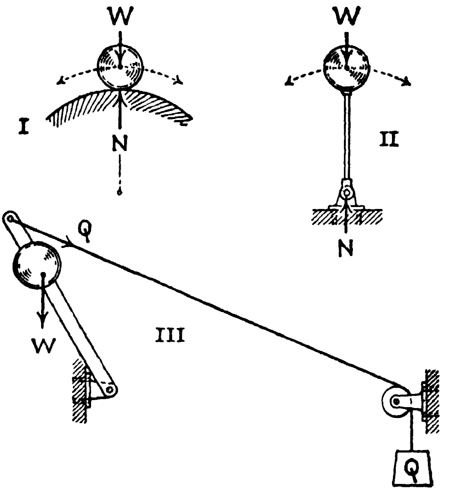


FIG. 151.

Figure 152 shows two cases of *neutral* or *indifferent* equilibrium. In either of these cases if motion through a short distance is caused by an additional force, the algebraic sum of the work done by the original forces is zero. All this, it should be noted, is for ideal conditions with no friction, and in the case of Fig. 152,II the weight of the rope is neglected. If this weight is considered this device is unstable.

In this discussion of static stability from the standpoint of work and energy, the displacements assumed were small but finite. For an infinitesimal displacement, the algebraic sum of the work done by all the forces of any system in equilibrium is zero, and the conditions for stable, unstable, and neutral equilibrium can be stated in precise mathematical form, as will be done later. There are also similar conditions of stability in dynamical and elastic systems described by such terms as *dynamic stability* and *elastic stability*. All three categories of stability have one thing in common: if the body or part in question is forced slightly away from its normal position or path by an additional force, it will return. Upon return, it may overrun but in this case it will return again and again.

When the stability of the system is wholly or partly due to forces like friction which can dissipate energy into heat, the statement above is not wholly exact, but it is certain that a static system which would be stable without friction would still be stable with friction. As a matter of fact, mechanical devices like those shown in Fig. 151 can all be made to stand in the positions shown by virtue of the friction inherent in them. Whether they could be made to stand if really perfect in form, without friction, and in a motionless world is a question which mathematics refuses to answer and which is beyond the possibility of an experimental determination.

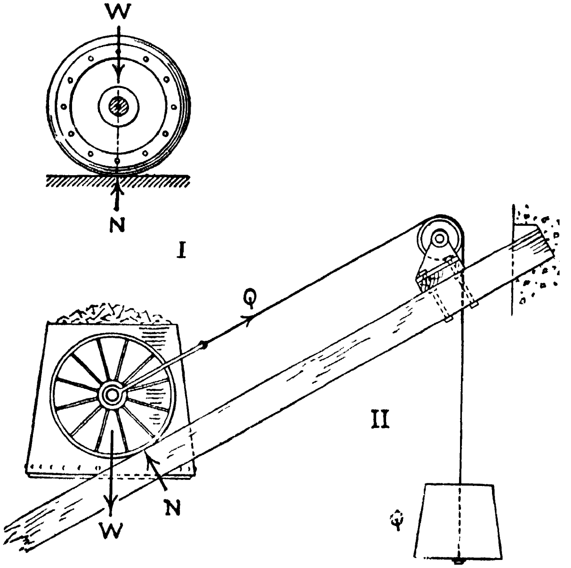


Fig. 152

Designers of structures, machines, and mechanical devices usually find their common sense a sufficient guide in determining static stability and need mathematical methods only in the more complicated cases.

73. Stability of Systems Having One Degree of Freedom. When a body or a system of bodies is so constrained that only one quantity need be given in order to determine its position completely, the body or system is said to have *one degree of freedom.* Nearly all the problems treated so far were of this kind. The arch of links of Fig. 138 was an exception. The toggle of Fig. 147 is a good example. When the physical dimensions of this device are known, its position can be established by the value of x, or it can just as well be established by the angle which either of the links makes with the horizontal, or even by y, but in this case it would need to be understood that the toggle pin is on a definite side of the center line, and y would not be a convenient variable to choose in determining the stability of the body. The bar of Fig. 146 is a simpler example, since any one of the three values x, y, or θ would serve equally well to determine its position, and it will be used as a first example.

If the bar of Fig. 146 is placed in some standard position, flat on the floor, for example, and then moved to some other position, the work required to make the move is called the *potential energy* of the bar in the second position. Suppose the bar weighs 60 pounds and is lying on the floor with the left end 10 feet from the wall and the right end against the wall. Suppose the force P is 20 pounds and is to remain constant in magnitude and direction throughout the motion, and suppose a force upward along the smooth wall is used to lift the right end until $x = 6$ and $y = 3$. The force at the right end would have to do work equal to $60 \times 3 - 20(10 - 6) = 100$ foot-pounds $= U$, the potential energy of the bar in the new position. This 100 foot-pounds is the work done on the bar by an *extra force* (perhaps hypothetical) in addition to the forces of the system under consideration. The work done on the bar by the *forces of the system under consideration* is clearly -100 foot-pounds, and an expression for this work is called the *energy function.* If the same standard position is agreed upon, the energy function Q is the equal and opposite of the potential function U, but since different standard positions may be used it is customary to write $U = C - Q$.

The student who has had a first introduction to potential energy is likely to protest here that the potential energy of the bar of the example is 180 foot-pounds, but this is just the potential energy of the *weight of the bar,* whereas the 100 foot-pounds is the potential energy of the system of forces P (acting as described), the two normal reactions (neither of which made any contribution), *and* the weight of the bar. The force P, acting as described, is called a *conservative* force. The student will get a better picture if he will imagine it to be applied by a string attached to the left end of the bar, passing along the floor, through a hole in the wall at the corner, over a smooth pulley, and down to a suspended mass of weight P. If the force P is exerted by a block nailed to the floor (a restraint) to hold the body in place or by a friction force at the lower end of the bar, the system is not a conservative system, and though the energy method serves to determine equilibrium it is useless in determining stability or instability.

Let U be the potential energy expressed in terms of a chosen variable θ. Then $U = f(\theta)$. Equilibrium exists if $dU/d\theta = 0$. The equilibrium is stable if, for that value of θ, $d^2U/d\theta^2$ is positive, or unstable if it is negative.

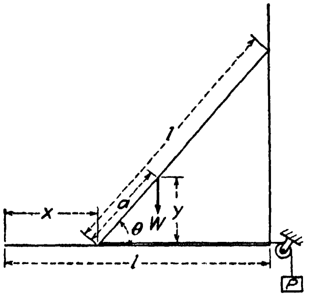


FIG. 153.

Example 1

In Fig. 153, find the force P to hold the bar in equilibrium at the angle θ and determine whether the equilibrium is stable.

Suppose the bar started lying on the floor and was moved to the inclined position by some external agency which did a quantity of work equal to U. The potential energy of the system in the inclined position is

$$U = Wy - Px \qquad (1)$$

From the figure, $y = a \sin \theta$, and $x = l\,(1 - \cos \theta)$. Substituting in Eq. (1),

$$U = Wa \sin \theta - Pl\,(1 - \cos \theta) \qquad (2)$$

$$\frac{dU}{d\theta} = Wa \cos \theta - Pl \sin \theta \; (= 0 \text{ if equilibrium exists}) \qquad (3)$$

$$\frac{d^2U}{d\theta^2} = -Wa \sin \theta - P \cos \theta \qquad (4)$$

Let $dU/d\theta = 0$, to find the position of equilibrium.

$$Wa \cos \theta - Pl \sin \theta = 0 \qquad W = \frac{Pl \tan \theta}{a} \qquad (5)$$

Equation (5) is exactly the same as the condition of equilibrium in Eq. (7) of Example 1 of Art. 71. When $\tan \theta$ is Wa/pl, as it must be for equilibrium from Eq. (5), the right-hand member of Eq. (4) is a negative quantity, indicating that the equilibrium is unstable.

If the work function instead of the potential function had been used in Eq. (1) (taking the same position as standard), the equation would have been written $Q = Px - Wy$. Every equation in the solution (Q replacing U) would be the same except that all signs would be changed. The same criterion for stability would have been used ($dQ/d\theta$ would have been equated to zero), but $d^2Q/d\theta^2$ would have turned out positive, indicating instability. The student must always be very clear as to which function he is dealing with, or his classification of the stability may be wrong. Otherwise, it usually makes little difference which of the functions is used.

If the body whose equilibrium is being considered is a bead on a smooth wire bent into a curve lying in a vertical plane, the physical significance of the propositions becomes clear and their validity obvious. The smooth wire can exert on the bead only a force normal to the wire, and the bead cannot be in equilibrium at any point where this normal force is not vertically upward, balancing the downward force W. Such points are shown at A, B, C, D, and E of Fig. 154. Let the equation of the curve of the wire be $y = f(x)$. The potential energy of the bead (measured above that which it has on the X axis) is

$$U = Wy, dU/dx = (W\,dy)/dx$$

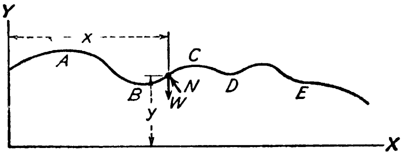


FIG. 154.

At the points where the wire is horizontal, dU/dx is zero, and these are the positions of equilibrium.

If the bead were displaced slightly from the position A or C it would slide downhill, as these are positions of unstable equilibrium. If it were displaced from B or D it would have to be pushed uphill, and if released it would return to these stable positions. The student will readily see that A and C represent positions of maximum potential energy, and he may remember that for these positions d^2y/dx^2 is negative, while B and D are minima with the rate of change of slope positive.

Position E represents a position of equilibrium at which the rate of change of slope is zero. This position is a position of unstable equilibrium because the bead, if displaced toward the left, will return, overrun, and proceed toward the right, just as though it had been displaced toward the right in the first place. If the bar of Example 1 were on a rough floor and just on the point of slipping, a small displacement of the bottom toward the left would cause it to fall, but a small displacement toward the right would not, so it is not analogous to the bead at point E on its wire. The friction is not a conservative force and it has already been ruled out of this argument.

Example 2

A small mass weighing W lb hangs by a light rod from a smooth hinge and is subjected to a steady horizontal wind force of P lb. Find the angle θ (Fig. 155) for stability and determine whether the equilibrium is stable or unstable.

The work required to bring the rod from the vertical downward position to the position shown in the figure (the wind blowing all the while) is

$$U = Wl(1 - \cos\theta) - Pl\sin\theta \tag{1}$$

$$\frac{dU}{d\theta} = Wl\sin\theta - Pl\cos\theta \tag{2}$$

For equilibrium,

$$Wl\sin\theta - Pl\cos\theta = 0 \qquad \theta = \tan^{-1}\frac{P}{W} \tag{3}$$

$d^2U/d\theta^2 = Wl\cos\theta + Pl\sin\theta$ is positive when θ is in the first quadrant, as shown in Fig. 155, and so this position is stable. The other position is 180° away and is unstable.

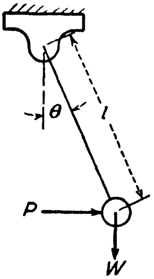


FIG. 155.

When the position of equilibrium has been determined by the method of virtual work (and the system is conservative), it is obviously easy to make the final step and determine the stability by a second differentiation. But when the position of equilibrium has been determined in some other way, it is usually convenient to assume a *virtual displacement* and find the work done (by some additional force) to cause this displacement. The expression for this work can be written in the form of a series in powers of the variable chosen to designate the position of the body as, $A + B\theta^1 + C\theta^2 + D\theta^3 + \cdots$. If B is not zero a mistake has been made in writing the equation or the position of the body was not one of equilibrium. If the squared term is positive the equilibrium is stable, and if it is negative the equilibrium is unstable.

This procedure differs only in form from the previous one. The total work to displace the body the infinitesimal distance is an infinitesimal, and the terms of the series above are called *small quantities of the first order*, *small quantities of the second order*, etc., depending on the exponent of the term. The force system in question must still be conservative. The student must be on guard, for some authors prefer to express the work done *by the forces of the system* during the small displacement, and in this case a negative sign on the small quantity of the second order is the mark of stability. An important problem in dynamics is that of determining the time for a body slightly displaced from a position of equilibrium to return to that position. In this problem the small quantity of work of the second order is related to the velocity of the body on its return and to the time of return in a simple way leading to an easy solution of the problem.

Example 3

With the pendulum of Example 2 standing in the lower position of equilibrium, find the expression for the work required to displace it slightly from this position.

The energy required could be found by substituting $\theta + \partial\theta$ for θ in Eq. (1) of Example 2 and subtracting therefrom the expression for U of that equation. It is better to observe that the resultant of P and W, $\sqrt{P^2 + W^2}$, at the position of equilibrium is downward and to the right in the direction of the rod. Then if the rod is turned through a small angle $\partial\theta$, the point of application of the resultant is moved a distance $l(1 - \cos\partial\theta)$ in the direction opposite to the direction in which the resultant acts. (The motion of this point in the direction perpendicular to the direction of action of the resultant involves no work and so need not be considered here.) Therefore, the work required to effect the motion is $\sqrt{P^2 + W^2}\,l(1 - \cos\partial\theta)$. $\cos\partial\theta$, expanded into a power series, becomes $1 - \dfrac{(\partial\theta)^2}{2!} + \dfrac{(\partial\theta)^4}{4!} \cdots$ and the expression for the work is $\sqrt{P^2 + W^2}\,l\left[\dfrac{(\partial\theta)^2}{2!} - \dfrac{(\partial\theta)^4}{4!} \cdots\right]$. There is no small quantity of the first order and the small quantity of the second order is positive, so there exists in this position stable equilibrium.

Problems

1. The pendulum of Example 2 has a constant wind force P acting vertically upward. Find the positions of equilibrium and the condition under which each is stable.
   Ans. Vertically downward and vertically upward. The second position is stable if $P > W$.
2. In Example 2 of Art. 71, assume the force P as constant and always horizontal. Set up the potential energy expression, assuming the lower link horizontal in the initial position. Solve for the condition of equilibrium and show that the equilibrium is unstable.
3. A straight bar of length l and weight W, with its center of mass at a distance a from the lower end, rests on a smooth horizontal floor and leans over the edge of a smooth wall. The vertical height from the floor to the point of contact with the wall is h. A rope attached to the bottom of the bar runs horizontally over a smooth pulley and supports a mass of P lb. Find the equation of equilibrium by the method of virtual work. *Ans.* $Wa(\cos\theta - \cos^3\theta) = Ph$.
4. Solve Prob. 3 by trial and error for the following cases:

l, ft	a, ft	h, ft	W, lb	P, lb
20	10	12	15	4
16	10	12	30	8
20	10	12	20	5

5. Solve Prob. 15 of Art. 56 by the method of virtual work and determine the condition of stability of the system.

6. Figure 156 shows a uniform bar with a smooth roller at the lower end running on a circular curve of radius R. The bar is provided with a slot which slides on a smooth fixed pin at a distance d above the center of the curve. Under what condition will the bar be stable in the vertical position? *Ans.* $\frac{l}{2} < \frac{(R+d)^2}{R}$.

7. A uniform bar of length L and vertical thickness d is balanced on a horizontal cylinder of radius R. Find the relation of d and R for stability.

8. A rod of length l (Fig. 157) with its center of mass at a distance a from one end slides inside a smooth hemispherical bowl of radius r. Find the position of equilibrium if l is less than $2r$.

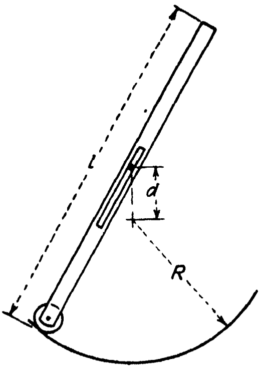


FIG. 156.

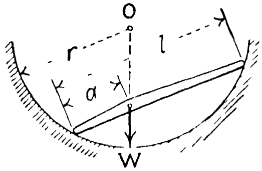


FIG. 157.

It is evident that the path of the center of mass is a circle with its center at the center of the sphere. Equilibrium exists when the center of mass is at its lowest point directly under the center of the sphere. In this position, find the angle which the bar makes with the horizontal.

9. Solve Prob. 8 when $l = 6$ in., $a = 2$ in., and $r = 4$ in.

Ans. The bar makes an angle of 20°42′ with the horizontal.

10. A bar AB (Fig. 158) is 8 ft long and has its center of mass 5 ft from A. The end B rests against a smooth vertical wall. The end A is supported by a cord, 10 ft in length, which is fastened to the vertical wall directly over the end B. Find the position of equilibrium. Solve by virtual work and check by the direction condition of equilibrium.

Ans. The bar makes an angle of 53°06′ with the vertical. Unstable.

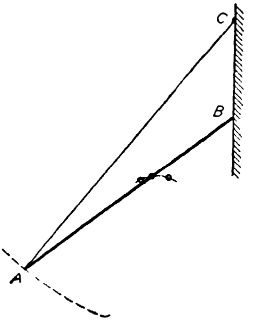


FIG. 158.

Figure 159 shows a simple pendulum displaced a small angle θ (exaggerated in the figure) from the position of equilibrium. The potential energy is $Wl(1 - \cos\theta)$ which becomes, on expanding and dropping terms higher than the square, $Wl\,\theta^2/2$. This expression occurs so often in energy equations that it is advisable to reduce it to a geometrical concept. In Fig. 159 the chord AB is equal to $l\theta$, θ being in radians and a small angle. The angle between the chord and the horizontal is $\theta/2$, and the height of the point B above A is

$l\theta^2/2$. This relation, together with the relation that $\theta = \sin\theta = \tan\theta$ for small angles, if powers above the second are neglected, greatly simplifies the work of solving many problems in stability and vibrations.

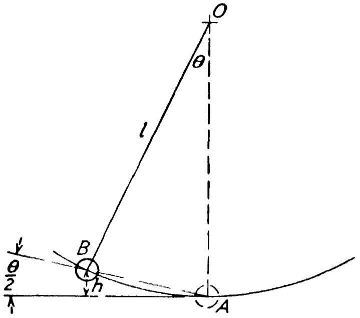


FIG. 159.

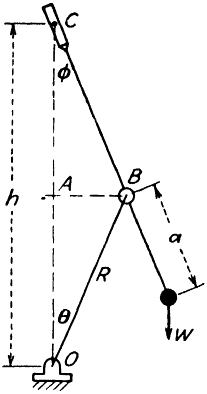


FIG. 160.

Problems

11. Figure 160 is the Schlick pendulum, a very useful device in vibration studies, which can be arranged to give a very slow vibration without excessive length. Find the relation of the factors R, h, and a for stability when the rods are vertical. (Rod CBW slides on the smooth fixed pin C directly above O.) When R is turned through the angle θ, B moves downward $R\theta^2/2$, BC turns through the angle ϕ, and W moves upward relative to B, $a\phi^2/2$. Also, $AB = R\theta = (h - R)\phi$. *Ans.* $a = (h - R)^2/R$.

12. Solve Prob. 6, making use of the approximate relations.

13. Another pendulum arrangement for producing slow vibrations in the metronome shown in Fig. 161. The pendulum is inverted and a spring arranged to restore it to place after a small displacement. If the spring is stretched or compressed a distance x, the restoring force is Kx, the K being called the *constant* of the spring. The work done to stretch the spring a distance a is $\int_0^a Kx\,dx$. When the rod of the metronome is turned through a small angle θ, the spring is stretched or compressed a distance $b\theta$, the small vertical movement of the end of the spring being neglected. Find K in terms of b, h, and W for stability of the metronome, if h is the height to the center of gravity.

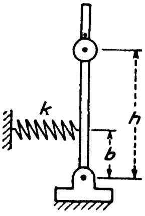


FIG. 161.

14. A picture hangs by a cord of length $2a$ passing over a smooth nail and attached to the upper corners A and B of the frame. If the nail is at a distance b above the center of the line AB and the center of gravity is at a distance c below the same point, what is the required relation of a, b, and c for stability with AB horizontal? *Ans.* $(a^2 - b^2) < bc$.

15. If the cord of Prob. 14 is too short, at what angle will AB stand with the horizontal?

Ans. $\tan^{-1}\dfrac{ac}{\sqrt{(a^2 - b^2)^2 - b^2c^2}}$.

16. The slender bar magnet has one end resting on a smooth floor and the other end attracted by the pole of another bar magnet (Fig. 162). The attraction between the two poles is C/y^2. Neglect the repulsion of other poles. Find y at the equilibrium point and determine stability.

Ans. $y = \sqrt{2C/W}$. Unstable.

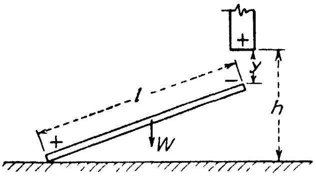


FIG. 162.

17. In Prob. 16 if h is greater than l the slender magnet will be in equilibrium standing vertically under the attracting pole. Determine the limits of y for stability in this position.

Ans. $\sqrt{2C/w} > y > \sqrt{C/W}$

CHAPTER 8

CONCURRENT, NONCOPLANAR FORCES

74. Resultant and Components. It is an experimental fact that the resultant of two concurrent, coplanar forces is represented by their vector sum. Starting from this fact, it has been shown that the resultant of any number of concurrent, coplanar forces may be represented by their vector sum. Finally, it has been proved that the direction and magnitude of the resultant of any number of nonconcurrent, coplanar forces may be determined from the vector sum of these forces.

The next problem for consideration is that of concurrent, noncoplanar forces. Figure 163 shows two forces **P** and **Q** in a vertical plane, and a single force **H** in the horizontal plane. The resultant of **P** and **Q**, or of any number of forces in the same plane, is the single force **R** in that plane. This resultant **R** and the force **H** intersect at the point of application. They lie in the inclined plane OAB of Fig. 163,II. Since **R** and **H** are in a plane, their resultant is their vector sum in that plane. The force **H** as well as the force **R** may be the resultant of several forces in its plane. In Fig. 163, the original planes are normal to each other. There is nothing in the proof which is limited to this position of the planes. They may have any angle with each other.

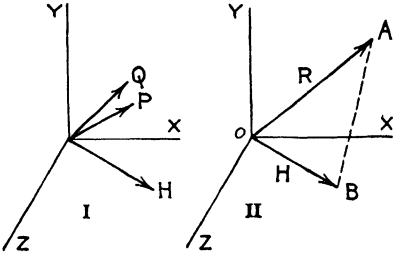


FIG. 163.

The resultant of concurrent forces in space may be represented by their vector sum. The forces which make up the resultant are its *components.*

75. Resolution of Noncoplanar Forces. A vector in a given plane may be specified numerically by two quantities. These may be its length and its angle with some axis, which correspond with polar coordinates, or its components along two axes, which correspond with Cartesian coordinates.

To specify fully a vector in space, three quantities are required.

These may be its length and two angles, or its components along three axes which are not all in the same plane.

In Fig. 164, a vector OP of length l is drawn as the diagonal of a rectangular parallelepiped. Three edges of this parallelepiped are taken as the axes of coordinates. The vector makes angles of α, β, and γ with the X, Y, and Z axes, respectively. These are the *direction angles* and their cosines are called the *direction cosines.*

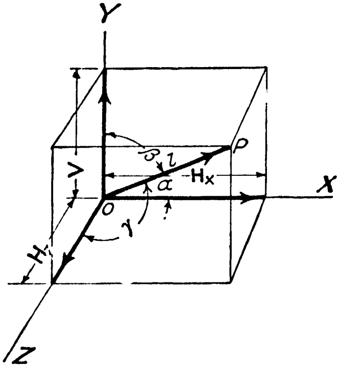


FIG. 164.

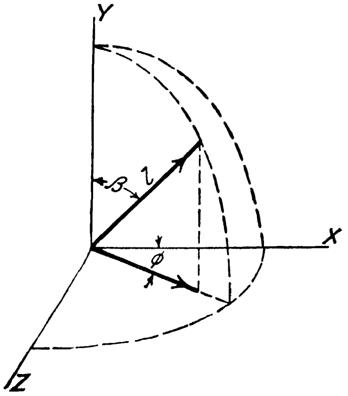


FIG. 165.

$$H_x = l \cos \alpha \tag{1}$$
$$V = l \cos \beta \tag{2}$$
$$H_z = l \cos \gamma \tag{3}$$

Since

$$l^2 = H_x^2 + V^2 + H_z^2 = l^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$$
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \tag{4}$$

Equation (4) is a fundamental formula of solid analytic geometry.

Figure 165 shows another way of expressing angles in space. The vector of length l makes an angle β with the Y axis, while the plane which passes through the vector and the Y axis makes an angle ϕ with the XY plane. This is equivalent to *colatitude* and *longitude* on a sphere. The Y axis may be regarded as the polar axis of the sphere. The angle β is equivalent to the colatitude and the angle ϕ is equivalent to the longitude. These coordinates are called *spherical coordinates.*

If H is the component in the XZ plane and V is the component parallel to the Y axis (Fig. 166),

$$H = l \sin \beta \tag{5}$$
$$V = l \cos \beta \tag{6}$$

If H_x is the component along the X axis and H_z is the component along the Z axis,

$$H_x = H \cos \phi = l \sin \beta \cos \phi \tag{7}$$
$$H_z = H \sin \phi = l \sin \beta \sin \phi \tag{8}$$

Figure 167 illustrates the graphical method of finding the components of a vector when the directions are given in spherical coordinates.

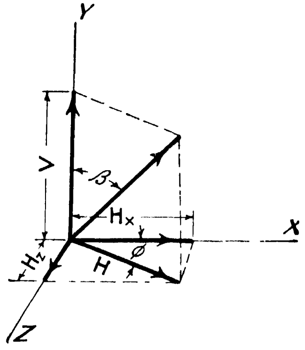


FIG. 166.

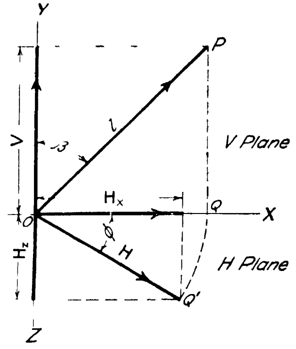


FIG. 167.

Problems

1. A force of 50 lb makes an angle of 60° with the vertical and an angle of 45° with a horizontal axis east. If the force is south of the vertical plane through the east axis, find the angle which it makes with the south horizontal axis. Find the components. *Ans.* 60°, 25 lb up; 35.35 lb east; 25 lb south.
2. A force of 50 lb makes an angle of 60° with the vertical, and an angle of 50° with the east horizontal axis. Find the angle with the south horizontal axis.

SUGGESTION: $$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

3. A force of 40 lb makes an angle of 50° with the vertical in a vertical plane which is north 25° east. Find its components. *Ans.* 25.71 lb; 27.77 lb; 12.95 lb.
4. In Prob. 3, find the direction cosines. *Ans.* 0.6428; 0.6943; 0.3237.

In actual machines and structures, linear dimensions are more likely to be used to designate space relations than are angles or coordinates based on any type of angular measurement. The direction cosines can usually be found from the dimensions without translating into angles in degrees and minutes. It is well known that the diagonal of a rectangular parallelepiped is the square root of the sum of the squares of the three sides. The length of any side divided by the length of the diagonal is the direction cosine of the diagonal with respect to that side.

Example

Figure 168 shows a 260-lb force acting at corner O toward a point 12 ft above the floor, 3 ft south of the north wall of a room, and 4 ft east of the west wall. The three dimension lines in the figure, the three solid lines representing the corners of the room, and the additional dotted lines form the parallelepiped *Oabcdefg*. The length of the diagonal is 13 ft.

The vertical component of the force is $^{12}/_{13} \times 260 = 240$ lb.

The east horizontal component is $^{4}/_{13} \times 260 = 80$ lb.

The south horizontal component is $^{3}/_{13} \times 260 = 60$ lb.

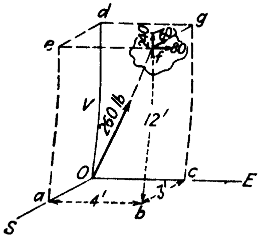


FIG. 168.

The resultant of these two horizontal components is in the direction Ob. The length Ob is 5 ft, the cosine of $fob = ^{5}/_{13}$. The horizontal component is then $^{5}/_{13} \times 260 = 100$ lb. If convenient in any solution, the 260-lb force can be removed and these three mutually perpendicular components (called a *triad*) placed anywhere on its line of action. The figure shows them placed at f. They can just as correctly be placed at O or at a point 12 ft above, 4 ft east, and 3 ft south of The student must exercise care not to use the force and the triad in any one equation and not to write an equation assuming part of the triad at one point and the remainder at another.

Problems

5. A mass of 51 lb is hung on a rope which runs over a smooth pulley and is attached to a fixed point which is 8 ft lower than the pulley, 9 ft west, and 12 ft south. Draw a rectangular parallelepiped, similar to Fig. 164, and calculate the component acting on the fixed point.

Ans. 24 lb up; 36 lb north horizontal; 27 lb east horizontal.

6. What angle does the vertical plane through the rope of Prob. 5 make with the north?

7. A horizontal trapdoor, 12 ft long by 8 ft wide, is lifted by a rope attached to one corner. The rope passes over a pulley which is 10 ft above the center of the door. When the vertical component of the pull is 40 lb, what is the component parallel to the end, the component parallel to the side, and the tension in the rope? Make a space parallelepiped and a force parallelepiped and calculate the horizontal components by proportion.

8. A light rod 8 ft long is placed with one end on a horizontal floor 4 ft from one smooth vertical wall and 2 ft from another smooth vertical wall at right angles to the other. It leans into the corner between the walls and a vertical load downward of 100 lb is applied to the upper end of the rod. Replace the force in the rod which holds the 100-lb load by a triad at the floor. Does this triad show what the floor is doing to the rod or what the rod is doing to the floor? If friction is depended on to keep the rod in place, what is the necessary coefficient at the floor? *Ans.* 0.674.

76. Calculation of Resultant. To find the resultant of a number of noncoplanar forces, it is best to resolve each force along each of three axes at right angles to each other, to add the components along each axis to get the component of the resultant along that axis, and finally to combine these components to form a single force. The *components* of the resultant are the edges of a rectangular parallelepiped of which the resultant is the diagonal.

Example

Find the resultant of the following three forces. 10 lb at 38° with the vertical in a vertical plane which is north 25° east, 12 lb east of north at an angle of 45° with the vertical and an angle of 60° with the north horizontal, and 13 lb applied by means of a rope which passes over a pulley 4 ft east, 3 ft south, and 12 ft above the point of application of the forces

Force	Components		
	Up	North	East
10	7 880	5 580	2 602
12	8 485	6 000	6 000
13	12 000	−3 000	4 000
	28 365	8 580	12 602

If ϕ is the angle which the vertical plane of the resultant makes with the north vertical plane,

$$\tan \phi = \frac{12\ 602}{8\ 850}$$

$$\phi = 55°45'$$

If H is the resultant of the two horizontal components,

$$H = \frac{12\ 602}{\sin \phi} = 15\ 245 \text{ lb}$$

If θ is the angle which the resultant makes with the vertical,

$$\tan \theta = \frac{15\ 245}{28\ 365} \qquad \theta = 28°16'$$

$$R = \frac{28\ 365}{\cos \theta} = 32\ 20 \text{ lb}$$

The resultant is a force of 32 20 lb at an angle of 28°16′ with the vertical in a vertical plane which is north 55°45′ east

Problems

1. Find the resultant of 25 lb at an angle of 30° with the horizontal in a vertical plane which is north 20° east and a force of 29 lb applied by a rope which passes over a pulley 16 ft west, 12 ft north, and 21 ft above the point of application of the two forces
   Ans. 47 35 lb at 45°02′ with the horizontal in a vertical plane north 14°53′ west.

2. *What angle does the resultant of Prob. 1 make with the west horizontal axis and with the north horizontal axis?* *Ans.* 79°33'; 46°55'.

77. Equilibrium of Concurrent, Noncoplanar Forces.

The condition of equilibrium for concurrent, noncoplanar forces is the same as for concurrent, coplanar forces. The force polygon must close; the resultant of all the forces must be zero. Since the force polygon is not all in one plane, three components are required to specify fully some of the forces. There are three unknowns and three independent equations. The sum of the components along any axis is zero. In order that the equations may be independent, one of these axes must have a component perpendicular to the plane of the other two.

In order that four forces may be in equilibrium, the resultant of two of them must lie in the plane of the other two. It is often desirable to replace two forces by their resultant at the intersection of their plane with the plane of the other two forces. The problem then becomes one of concurrent, coplanar forces. After this unknown resultant has been found, its components may be computed as a second problem of coplanar forces.

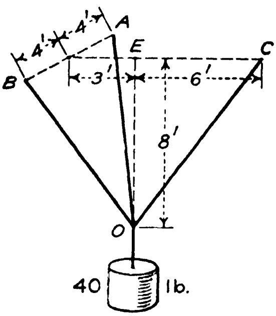


FIG. 169.

Example 1

A mass of 40 lb is supported by three ropes. The ropes are fastened to a horizontal ceiling 8 ft above the body at points A, B, and C. If E is a point in the ceiling directly over the body, the point A is 3 ft north and 4 ft east of E; the point B is 3 ft north and 4 ft west of E; the point C is 6 ft directly south of E. Find the tension in each rope.

A horizontal resolution perpendicular to the vertical plane ECO of Fig 169 eliminates the force in OC. The ropes OA and OB make equal angles with the axis of resolution. If the forces in the ropes are represented by A, B, and C, respectively,

$$A = B$$

By a vertical resolution,

$$2A \times \frac{8}{\sqrt{89}} + 0.8C = 40$$

By a horizontal resolution parallel to EC,

$$2A \times \frac{3}{\sqrt{89}} - 0.6C = 0$$

$$A = \frac{5\sqrt{89}}{3} = 15.72 \text{ lb}$$

$$C = 16.67 \text{ lb}$$

Example 2

In Fig. 169, let *B* be 3 feet west of the plane *ECO*, while the remaining data are not changed. Find the tension in each rope.

This example may be solved by three resolutions which give three equations and three unknowns, or by replacing the forces *A* and *B* by their resultant and solving two problems of concurrent, coplanar forces. The resultant of *A* and *B* on the intersection of the planes *OBA* and *OEC* may be represented by *T*. Vertical and horizontal resolutions in the plane *OEC* give

$$0.8C + \frac{8T}{\sqrt{73}} = 40$$

$$0.6C - \frac{3T}{\sqrt{73}} = 0$$

$$C = {}^{50}\!/_{3} = 16.67 \text{ lb}$$

$$T = \frac{10\sqrt{73}}{3} = 28.48 \text{ lb}$$

Resolutions parallel and perpendicular to *T* in the plane *OBA* give

$$\frac{\sqrt{73}A}{\sqrt{89}} + \frac{\sqrt{73}B}{\sqrt{82}} = T$$

$$\frac{4A}{\sqrt{89}} - \frac{3B}{\sqrt{82}} = 0$$

$$B = \frac{40\sqrt{82}}{21} = 17.25 \text{ lb}$$

$$A = \frac{10\sqrt{89}}{7} = 13.48 \text{ lb}$$

Problems

1. A weightless bar 11 ft long stands on a horizontal floor and leans against the intersection of two smooth vertical walls. The lower end of the bar is 2 ft from one wall and 6 ft from the other. A load of 180 lb is hung by a vertical string attached to the upper end of the bar. Find the force in the bar and the forces which the walls exert against the upper end of the bar. Find the friction force necessary to hold the lower end of the bar from slipping on the floor.

   Ans. Force in the bar = 220 lb compression.
2. The 200-lb load in Fig. 170 is supported by the two ropes *BA* and *CA* and by the horizontal force *P* perpendicular to the line *BC*. Find *P* and the forces *AB* and *AC*. *Ans.* *P* = 150 lb.

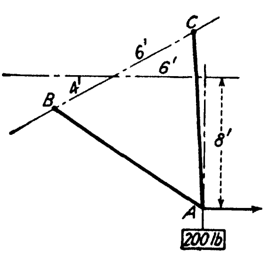


FIG. 170.

3. The bars *DA* and *CA* in Fig. 171 are horizontal. Find the forces in the three bars due to the 2000-lb load hung from *A*.
4. A weightless bar, 10 ft long, rests on a horizontal floor and leans against a smooth vertical wall. The lower end of the bar is 4 ft from the wall and is

held from slipping by a peg in the floor. The upper end of the bar is 3 ft to the right of a vertical plane perpendicular to the wall and passing through the lower end of the bar. A horizontal rope, parallel to the wall, is attached to the upper end of the bar. Find the compression in the bar, the tension in the rope, and the pressure against the wall when a load of 80 lb is placed on the top of the bar.

Ans. $160\sqrt{3}/3 = 92.37$ lb; 27.71 lb; 36.95 lb.

5. Solve Problem 4 if the left end of the rope is lowered to 20° below the horizontal.

Ans. 105.70 lb; 33.74 lb; 42.28 lb.

6. Solve Problem 4 if the rope is horizontal and is attached to a point 2 ft in front of the wall and 5 ft to the left of the top of the bar.

Ans. 92.37 lb; 29.84 lb; 25.87 lb.

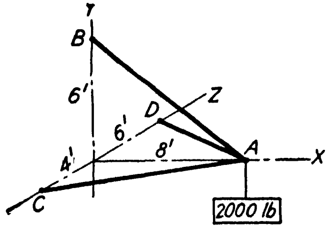


FIG. 171.

7. In Fig. 169, let A be 6 ft east and 5 ft north of E, and let B be 4 ft west and 3 ft north of E. The remaining data are not changed. Find the tension in each rope by the method of Example 2.

Ans. $C = 190/9.8 = 19.39$ lb; $A = 13.69$ lb; $B = 17.33$ lb.

8. A weightless bar OC, 13 ft long, has one end C pivoted against a vertical wall and the other end O elevated 5 ft above the level of C in a vertical plane through C perpendicular to the wall. A rope OB is attached to the wall at B, which is 5 ft above the level of C and 4 ft to the right of the vertical line through it. A second rope OA is attached to the wall 5 ft above the level of C and 9 ft to the left of the vertical line through it. Find the tension in each rope and the compression in the bar when a load of 130 lb is hung at O.

Ans. $C = 338$ lb; $B = 227.7$ lb; $A = 120$ lb.

78. Moment about an Axis. The moment of a force about an axis is the product of the component of the force in a plane perpendicular to the axis multiplied by the shortest distance from the line of the force to the axis. In Fig. 172, the force is **P** and the axis is OZ. The XY plane is perpendicular to the axis. The force **P** may be resolved into two components at right angles to each other. One component, which is not shown in Fig. 172, is parallel to the axis OZ. The other component is **P**′ in the plane perpendicular to the axis. The distance d from the axis to the line of the component **P**′ is the effective moment arm.

$$M = P'd$$

In Art. 25, the moment of a force was said to be about a point in its plane. It is better to consider the moment to be about an axis which passes through the point and which is perpendicular to the plane.

In the case of the moment of a force about a point in its plane, the apparent arm was drawn from the origin of moments to the line of the force; then this arm was multiplied by the component of the force perpendicular to it. Likewise, in the case of the moment of a force with respect to any axis, a plane may be passed through the axis of moments intersecting the line of force at any convenient point. The component of the force normal to this plane multiplied by the perpendicular distance from the point of intersection to the axis gives the required moment.

In Fig. 172, the plane *OXZ* passes through the axis *OZ*. The line of force may be extended until it intersects this plane at *C*. The moment of the force about *OZ* is the product of the vertical component $\mathbf{P}_v$ multiplied by the distance from *C* to the axis. The other components of the force **P** at this point lie in the plane *OXZ* and either intersect the axis or are parallel to it. Consequently, only the vertical component $\mathbf{P}_v$ has moment about the axis.

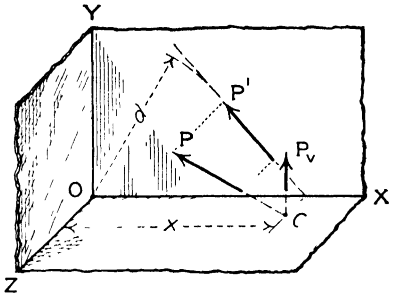


FIG. 172.

The line of the force $\mathbf{P}'$ might be extended in the opposite direction until it intersects the vertical plane *OYZ*. The horizontal component of the force at this point of intersection multiplied by the vertical distance from the point to the axis gives the moment. Any plane through the axis may be used in this way. The choice of a plane depends upon the difficulty of finding the component of the force perpendicular to it, and of finding the distance from the point of intersection to the axis.

Example

In Fig. 169, find the moment of the force in *OC* with respect to the line *AB* as an axis. When the horizontal plane *ABC* is considered as turned about the axis by the force in *OC*, the apparent moment arm in the plane is 9 ft in length. If *C* is the magnitude of the force in *OC*, its vertical component is 0.8*C*.

$$M = 0.8C \times 9 = 7.2C$$

The vertical plane through the axis may be considered as turned by the horizontal component of *C*. To find the apparent moment arm, the line *CO* is extended beyond *O* until it intersects this vertical plane. By similar triangles, this intersection is found to be 12 ft from *AB*. The horizontal component of the force is 0.6*C*.

$$M = 0.6C \times 12 = 7.2C$$

Since the plane *ECO* of Fig. 169 is perpendicular to the axis *AB*, the force in *OC* has no component parallel to this axis, and the moment may be calculated by multiplying the entire force by the perpendicular distance from the middle of the axis *AB* to the line *OC*. This distance is $0.8 \times 9 = 7.2$ ft, and the moment $= 7.2C$.

Problems

1. *A rectangular door, 5 ft wide and 8 ft high, stands in a north and south vertical plane with the hinges at the north edge. It is turned by a cord fastened at the south edge 6 ft above the bottom. This cord runs over a pulley which is 6 ft west of the hinges and 6 ft above the level of the bottom of the door. Find the moment about the hinges when a load of 20 lb is hung on the rope.* *Ans.* $M = 5 \times 6/\sqrt{61} \times 20 = 76.82$ lb-ft.
2. Solve Prob. 1 if the pulley is 3 ft above the level of the bottom of the door and the remaining data are not changed. *Ans.* $M = 71.71$ lb-ft.
3. In Fig. 168, find the moment of the 260-lb force with respect to the line *ed*. *Ans.* With the triad at *f*, $240 \times 4 = 960$ lb-ft; with the triad at *O*, $80 \times 12 = 900$ lb-ft.
4. Solve Prob. 3 with the triad at the mid-point of the line *Of*.
5. Find the moment of the 260-lb force in Fig. 168 with respect to the diagonal *eb*. *Ans.* $4 \times 60/\sqrt{160}$ lb-ft.

   SUGGESTION: To find the moment of a force in space with respect to a line in space, it is often convenient to resolve the force into an orthogonal triad at some point conveniently located on the line of action of the force, making one component parallel to the axis and a second component intersect the axis. The moment is the third component multiplied by the distance from the location of the triad to the axis about which moment is taken.
6. A force of 58 lb acts along the line which passes through the point $x = 15$, $y = 16$, $z = 0$; and the point $x = 3$, $y = 0$, $z = 21$. Find the moment of this force about the X-axis. Use the XY plane and check by the XZ plane.
7. Solve Prob. 3 for the moment about the line $y = 2$, $z = 0$.
8. Solve Prob. 3 for the moment about the line $4x = 3y$, $z = 0$. *Ans.* 100.8 lb-ft.

79. Equilibrium by Moments. If a body is in equilibrium under the action of a number of *concurrent, noncoplanar* forces, the sum of the moments of these forces about any axis is zero. If a plane is passed through the axis of moments and the point of application of the forces, the moment of each force will be the product of its component perpendicular to the plane multiplied by the distance of the point of application from the axis. Since the moment arm is the same for all the concurrent forces, the sum of the moments is equivalent to a resolution perpendicular to the plane. It is not necessary to use this plane in calculating the moments. Any convenient plane may be used, and different planes may be employed for the different concurrent forces.

Example 1

Solve Example 2 of Art. 77 by moments. Figure 173 shows the 40-lb load and the three unknown forces *A*, *B*, and *C* acting at *O*. At the points *A*, *B*, and *C* in the ceiling, these forces are shown again in the form of components. Only the magnitudes of *A*, *B*, and *C* are unknown so each triad involves only one unknown.

At C are the components, ~~0.6C south and~~ 0.8C vertically upward. At A is the triad $\frac{4}{\sqrt{89}}A$ to the east, $\frac{3}{\sqrt{89}}A$ north, and $\frac{8}{\sqrt{89}}A$ vertically upward, the fractions being the direction cosines of the force A with respect to the three axes chosen. A similar triad is set up at point B.

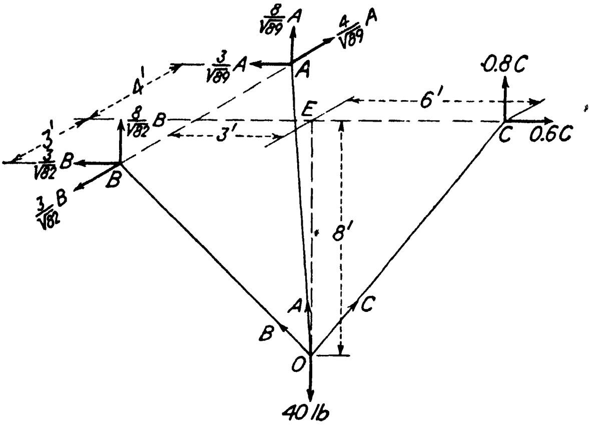


FIG. 173.

Moments about the line AB in the ceiling,

$$0.8C \times 9 = 40 \times 3 \tag{1}$$

Whence

$$0.8C = {}^{40}\!/_{3} \qquad C = 16.67 \text{ lb}$$

Moments about a line through B parallel to EC,

$$\frac{8}{\sqrt{89}}A \times 7 + 0.8C \times 3 = 40 \times 3 \tag{2}$$

Whence

$$A = 13.48 \text{ lb}$$

Moments about a line through A parallel to EC,

$$\frac{8}{\sqrt{82}}B \times 7 + 0.8C \times 4 = 40 \times 4 \tag{3}$$

Whence

$$B = 17.25 \text{ lb}$$

This solution as compared with Example 2 of Art. 77 saves little or no labor. That example reduced the simultaneous equations required from three to two by the artifice of introducing the resultant of A and B into the problem. At the cost of considerable geometric difficulty, any of the three unknowns can be found from a single equation of resolution by resolving in a direction perpendicular to the plane of the other two unknowns. In the above example, moments about the line

AB eliminated the unknowns *A* and *B* and gave an equation in *C*. In a similar manner, the example below solves for *B* by taking moments about the line *AC*. Figure 174 is a plan view of the points in the ceiling to show how the moment arms *BG* and *EF* are found. This example will be solved again by an elegant method in Art. 89.

Example 2

To find the vertical component of the force in *OB* (Fig. 173), *AC* is taken as the axis of moments. If V_b is the vertical component of *B*, V_b multiplied by the length *BG* is equal to 40 lb multiplied by the length *EF* (Fig. 174).

$$BG = 7\cos\theta$$
$$EF = FH\cos\theta = \tfrac{8}{3}\cos\theta$$
$$7V_b = \frac{8\times 40}{3}$$
$$V_b = {}^{320}/_{21}$$
$$B = \frac{320}{21}\times\frac{\sqrt{82}}{8} = 17.25\text{ lb}$$

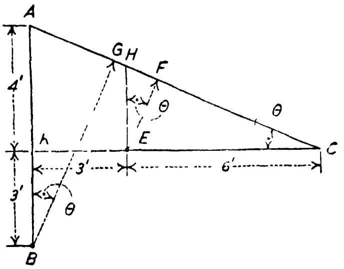


FIG. 174.

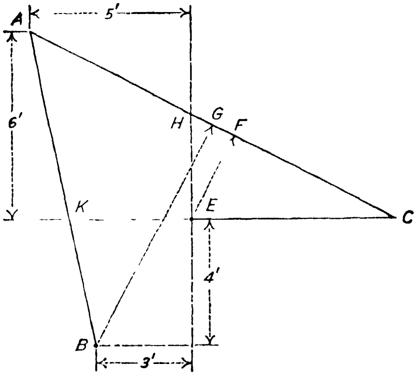


FIG. 175.

For a problem of concurrent noncoplanar forces, the basic equations are the three equations of resolution but, as is always true, any resolution equation can be replaced by a moment equation, so the problem can be solved by one moment and two resolutions, by two moments and one resolution, or by three moments. In case moment equations are substituted for resolution equations, there is always the danger of writing equations which are not independent, but this danger is negligible when the axis has been chosen *not at random but with a view to getting a simple equation involving the unknown sought.*

Problems

1. Find the force *A* of Example 1 by moments about the line *BC* of Fig. 174.
2. Solve Prob. 7 of Art. 77 by moments. Figure 175 is a plan view of the points

of attachment of the cords at the ceiling, showing EF and BC, the moment arms required for finding force B in a manner similar to that used in Example 2.

3. Solve Prob. 3 of Art. 77 by moments.
4. DEC is a vertical line on a wall with D 4 ft above E and C 5 ft below E. A is 6 ft to the right of D and B is 13 ft to the left of D. O is 12 ft from E on a line perpendicular to the wall at E. AO, BO, and CO are bars hinged together at O and hinged at A, B, and C. Find the forces in the three bars due to a horizontal force of 240 lb at O acting parallel to AB.
5. Solve Prob. 4 with the 240-lb force acting toward E.

80. Space Frameworks. A framework in space should, if the members are to be two-force members, have joints of the ball-and-socket type without friction. Of course no such ideal joint can be made and there would rarely be any reason to construct one even approximately except in functional machinery where considerable freedom of movement is required. In structural designs the members are often flexible enough so that the rigidity of the joints causes only minor secondary stresses. In this case such structures can be analyzed by the method of joints used in plane trusses. The problems of the preceding articles are typical of the analysis of a single joint of such structures.

The tetrahedron is the basic rigid element in a space framework, just as a triangle is the basic element in a plane. A single tetrahedron has four joints and six members, and for each tetrahedron added to the first there is one additional joint and three additional members, so that the number of members in a rigid space framework must be *six less than three times the number of joints.* Such structures (except in air and water craft) rest on some support such as the ground, and the ground, from anchorage to anchorage, may serve as one or more of the members and must be so counted in applying the rule.

Space frameworks which are statically determinate can be readily solved by the method of joints similar to that used in plane trusses, each joint being solved in turn with the values found from the solution of one joint becoming the known values in the next problem. In reality, plane trusses are merely elements in space frameworks, since they must always be braced in some way to secure stability in the plane. An ordinary through truss bridge has a plane truss on either side of the roadway with beams extending from the lower joints of one truss to the corresponding joints of the other. Over these beams run the stringers which support the floor but are not a part of the space truss. The diagonal bracing between the floor beams forms a part of the space truss as do the struts and diagonal bracing (sway bracing) between the top joints of the two plane trusses. Also, there is the "portal bracing" to complete the rigidity of the framework.

The method of sections in space frameworks constitutes a practical means of solving space-framework problems. This method is not elaborated in this text but is indicated in Probs. 16, 17, and 18 of Chap. 9. These problems refer back to Fig. 176.

Figure 176 shows a space framework from which the members forming the triangle *ABC* might well have been left out. With these members in place and with a vertical load at *H*, the ground reactions at *A*, *B*, and *C* are all vertical, and the external reactions on the framework constitute a system of parallel forces in space with three unknowns, a type which will be discussed in a later chapter. (If the external system of forces acting on a space framework is not statically determinate, the forces in at least some of the members of the framework will not be determinate.)

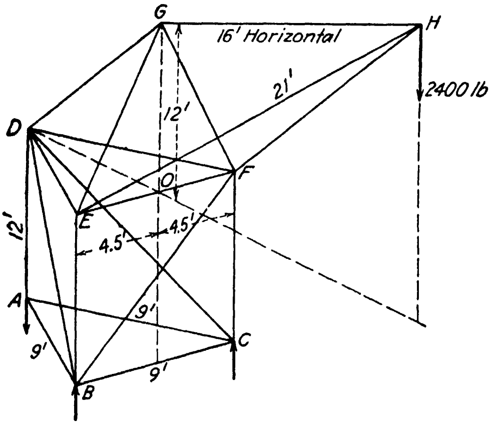


FIG. 176.

Figure 176 is symmetrical about a vertical plane through *DO*. The dimensions have been chosen to avoid the tedium inherent in the solution of a space framework so that the student can concentrate on the method rather than on the computation. Any joint can be solved, providing not more than three unknowns remain at the joint. When a joint has been solved, arrows should be placed at *both ends* of all the members connected to that joint, so that the number of unknowns at the remaining joints will be readily apparent.

Problems

1. Determine by the rule whether the framework of Fig. 176 is rigid and not redundant.
2. Begin at *H* and solve for the forces by the method of joints, using equations of moments or resolutions according to convenience.
3. Find the forces in the framework of Fig. 176 with a horizontal force of 900 lb at *H* acting in the direction *FE* (no vertical load).
4. Solve Prob. 3 if the horizontal force at *H* is 800 lb acting in the direction *GH*. (Answers to Probs. 2, 3, and 4 for the member *FC* are contained in the answer to Prob. 16, Art. 90, Page 193.)

CHAPTER 9

NONCONCURRENT, NONCOPLANAR FORCES

81. Definition. The designation *nonconcurrent, noncoplanar* indicates that the forces of the system do not lie in a single plane and are not concurrent at a single point. Such a system may be only apparently nonconcurrent and noncoplanar, since one or more forces of a coplanar system may be replaced by components lying in a plane intersecting the original plane of the system and thus give the revised system the appearance of being noncoplanar. For a general system of nonconcurrent, noncoplanar forces not in equilibrium, there is no direction in which the components of the system sum up to zero and no axis about which the sum of the moments of the system is zero. Such a system cannot consist of less than three forces. Since a line in space can be drawn to intersect any other two lines, an axis could always be drawn intersecting the lines of action of any two forces and a two-force system would have no moment about such an axis. Two of the three forces may constitute a couple, and when the third force is so placed that it is perpendicular to the plane of this couple, the combination of force and couple is said (rather academically) to be the resultant of the system.

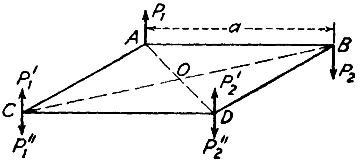


FIG. 177.

82. Resultant of Couples in Space. It has been shown that a couple in a plane can be replaced by another couple of the same magnitude and sign. The forces of the couple may have any magnitude and position and may be located anywhere in the plane of the original couple. It can readily be shown that a couple acting on a body in any plane can be replaced by an equal couple in any parallel plane without affecting the resultant of the force system. (This is not to say that the *stresses* within the body are unaffected.) Figure 177 represents a rectangle *ABCD* with two equal forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in opposite directions at *A* and *B*, forming the couple of magnitude *Pa*, in a plane perpendicular to the line *CA*. At *C* add two equal and opposite forces $\mathbf{P}_1'$, and at *D* add two more equal and opposite forces, $\mathbf{P}_2'$. All six forces are equal in magnitude. The two pairs added have zero resultants, so the system has not been changed. The resultant of P_1 and P_2' is an upward force **2P** at *O*, the center of the rectangle, and the resultant of $\mathbf{P}_2$ and $\mathbf{P}_1''$ is an equal downward force at the same point. These two resultants cancel. All forces can then be removed except $\mathbf{P}_1'$ and $\mathbf{P}_2''$, a couple equal to the original couple but moved to a parallel plane. The particular plane in which the couple acts is not an essential part of its effect. A line normal to the plane can be used to indicate the group of parallel planes, to any one of which the couple could be moved without changing its effect.

All the essential elements of the couple can be represented by a vector if the vector is drawn normal to the plane of the couple and given a magnitude equal (to some scale) to the moment of the couple represented. In the case of couples in space it is advantageous to do this. The direction assigned to the vector can be made to represent the sense of the couple (whether clockwise or counterclockwise) and in this text, as in most treatments of the subject, the right-hand screw convention is used. If an ordinary screw is imagined embedded in the plane of the couple and turned in the sense indicated by the couple, the arrow is pointed in the direction the screw would progress. That is, a clockwise couple is represented as going away from the observer and a counterclockwise couple is represented as coming toward him.

If two couples which are not in parallel planes are represented by vectors, the resultant of the two vectors represents the sum of the two couples. This proposition follows from the general rule of the addition of vector quantities, but for the benefit of the student to whom the proposition in this case is not obvious, the following proof is offered:

Figure 178,I represents two planes which intersect on the line *AB*. A counterclockwise couple of moment *Pa* is supposed to be acting in the plane *CAB* and a second counterclockwise couple *Qb* in the plane *DAB*. The couple *Pa* may be anywhere in its plane. One force $\mathbf{P}_2$ may be regarded as acting along the line *AB* and the second force $\mathbf{P}_1$ at a distance *a* from that line. The couple *Qb* may be replaced by a couple *Pc* in its plane, provided $Pc = Qb$. One force of this new couple may be regarded as acting in the line *AB* in a direction opposite to the force $\mathbf{P}_2$. The remaining force will then be in the plane *DAB* at a distance *c* from the line *AB*. The forces $\mathbf{P}_2$ and $\mathbf{P}_3$ acting along the line *AB* in opposite directions balance each other. The remaining forces $\mathbf{P}_1$ and $\mathbf{P}_4$ form a new counterclockwise couple. If *d* is the distance from $\mathbf{P}_1$ to $\mathbf{P}_4$, the moment of this couple is *Pd*. Figure 178,II is a space triangle in a plane perpendicular to the planes *CAB* and

DAB. The line *HL* of length *d* in this triangle is the distance between the forces of the resultant couple. Figure 178,III is a vector diagram. The vector **EF** of length *Pa* represents the original couple in the plane *CAB*. The vector **EF** is normal to this plane and, therefore, parallel to the plane of the triangle *HLK* and perpendicular to the line *HK*.

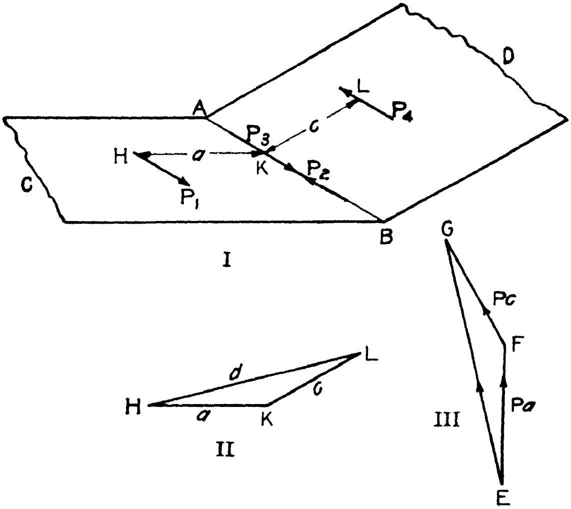


FIG. 178.

The vector **FG** is normal to the plane *DAB* and is normal to the line *KL* of the space triangle. Since

$$\frac{Pa}{Pc} = \frac{a}{c}$$

and since the angle at *F* is equal to the angle at *K*, the space diagram and the vector diagram are similar triangles. As the angle at *E* is equal to the angle at *H*, *EG* is normal to *HL*. Since the sides are proportional

$$\frac{EG}{Pa} = \frac{d}{a}$$

$$EG = Pd$$

If two couples are represented by vectors normal to their respective planes, the resultant of these couples is represented by the vector sum of the two vectors.

Example

A force of 12 lb is directed east in a horizontal plane. In the same horizontal plane 2 ft north a force of 12 lb is directed west. A horizontal force of 8 lb is *directed north. Vertically 4 ft over this force a horizontal force of 8 lb is directed south. Find the resultant couple.*

The moment of the forces in the horizontal plane is 24 lb-ft. This couple is represented by a vector 24 units long directed vertically upward. The second couple is represented by a vector 32 units long directed horizontally toward the east. The vector which represents either of these couples may be placed in any position. In Fig. 179 the vectors are shown in a vertical east and west plane. The resultant vector is 40 units at an angle of 53°08′ with the horizontal. A couple of 40 lb-ft in any plane perpendicular to this vector, turning counterclockwise to an observer east of and above the plane, is equivalent to the combination of the two original couples.

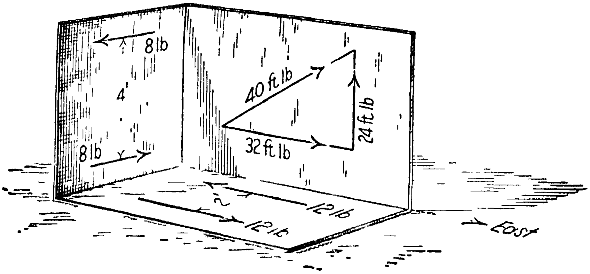


FIG. 179.

When the planes of the couples to be combined are not at right angles to each other, it is advisable to resolve each vector along the three coordinate axes and then combine the components in the same way as was done in the calculation of the resultant of noncoplanar forces.

Problems

1. A force of 20 lb acts from the point (0,0,4) toward (6,0,4), and a force of 20 lb acts from (5,3,0) toward (0,3,0). A force of 13 lb acts from (0,6,0) toward (5,6,12), and a force of 13 lb acts from (5,0,12) toward (0,0,0). Find the magnitude of the resultant couple and the direction of any line normal to its plane. *Ans.* 111.7 lb-ft. The normal makes an angle of 44°16′ with the Y axis in a plane through the Y axis at 22°37′ with the XY plane.

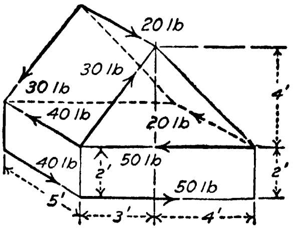


FIG. 180.

2. In Fig. 180, find the sum of the components of the moment vectors along each of the three axes.

Ans. 170 lb-ft about a vertical axis; 120 lb-ft about a horizontal axis toward the left; 100 lb-ft about a horizontal axis toward the front.

83. Resultant of Forces in Space. It was shown in Art. 53 that a force in a plane is equivalent to a force in the same plane at some other point plus a couple in the plane, provided the replacing force has the same magnitude and direction as that of the original force and the *couple has a magnitude equal to the moment of the original force with respect to a point on the line of action of the replacing force. In a like manner, a force in space may be moved from one line of action to another, provided there is added to the system one or more couples so that the moment of the new system of force and couples is, with respect to any axis, the same as was the moment of the original force.* Figure 181 shows three rectangular axes OX, OY, and OZ and a force passing through some point p whose coordinates are x, y, and z. The force has already been resolved into components **X, Y,** and **Z.** The component **Y** has been encircled to indicate its removal to O, where it is shown as a single-barbed arrow The original component **Y** had a moment about the X axis equal to $\mathbf{Y}z$, and this is shown by the double-barbed arrow on the X axis. This double-barbed arrow $\mathbf{Y}z$ represents a couple in the YZ plane (or in any plane parallel to plane YZ). The moment of the original component **Y** about the Z axis is $\mathbf{Y}x$ and the double-barbed arrow representing this couple is shown on the Z axis. Moment vectors pointing toward the positive ends of the axes are positive, according to the usual conventions. The student should now move the components **X** and **Z** to the origin and place on the figure the double-barbed arrows to represent the couples which must be supplied to compensate the moves. The figure will then show the triad of forces at the origin, and on each axis two couple vectors, in opposite directions and neither containing the letter designating that axis. For example, the y axis will show couple $\mathbf{X}z$ pointing upward and $\mathbf{Z}x$ downward. The force in Fig. 181 was in a direction such that all three components **X, Y,** and **Z** were in positive directions and it was formed into a triad at a point whose coordinates were all positive. The figure should help the student understand what is meant by the formulas for the moments about rectangular axes of a force in space:

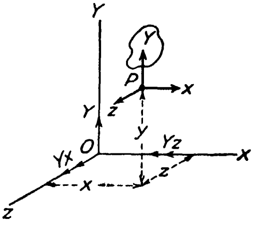


FIG. 181.

$$\left.\begin{aligned} M_x &= \mathbf{Z}y - \mathbf{Y}z \\ M_y &= \mathbf{X}z - \mathbf{Z}x \\ M_z &= \mathbf{Y}z - \mathbf{X}y \end{aligned}\right\} \qquad \text{Formula VI}$$

It is clear that each force of any system may be replaced by a triad at any chosen origin plus a triad of moment vectors along the same axes. Finally these two sets of triads can be composed each into a single vector, one representing the vector sum of all the forces and the other representing the cognate couple of the system of forces replaced by a ***single force at that chosen point. If some other point is chosen as the location of the resultant force, the force is unchanged, but the magnitude and direction of the resultant couple will be different.***

In the following example, the components and the moments of the various forces have been tabulated instead of being represented as vectors. This procedure is usually more convenient for actual computation.

Example

A rectangular parallelepiped, 15 by 8 by 6 in., has the 15- by 8-in. faces horizontal. It weighs 6 lb and has its center of gravity at the center. A horizontal force of 34 lb is applied at the top along the diagonal AC of Fig. 182. A force of 25 lb is applied at the right along the diagonal GB. A horizontal force of 24 lb, toward the front, is applied to the left face midway between the top and bottom. Replace these forces by a force at O and a couple.

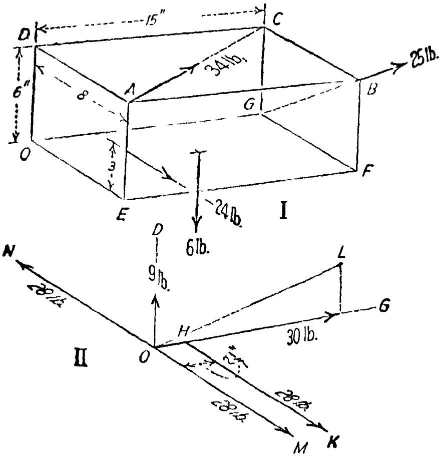


FIG. 182.

The edges OG, OD, and OE may represent the X, Y, and Z axes, respectively.

Force	Components			Moments		
	X	Y	Z	M_x	M_y	M_z
6	0	−6	0	24	0	−45
34	30	0	−16	−96	240	−180
25	0	15	20	0	−300	225
24	0	0	24	72	0	0
	30	9	28	0	−60	0

The resultant force is 42.01 lb at an angle of 12°22′ with the XZ plane in a plane *through the Y axis at 43°01′ with the XY plane. When the resultant force passes through O, the couple is 60 lb-in. clockwise as viewed from above.*

A couple of 60 lb-in. is equivalent to two forces of 28 lb each at $2\frac{1}{7}$ in. apart. On Fig. 182,II, ON and HK make the couple. The force ON balances the force OM and leaves the force HK. The forces which act on the parallelepiped are equivalent to a force of 28 lb parallel to the Z axis at a distance of $2\frac{1}{7}$ in. from O toward the right, and a force through O, which is the resultant of 30 lb along the X axis and 9 lb along the Y axis.

Problems

1. Replace the couple of the above example by two horizontal forces of 30 lb and show the position, magnitude, and direction of two forces which are equivalent to all the forces which act on the parallelepiped.
2. Can the couple in the above example be replaced by two vertical forces of 9 lb? Why?

84. Equilibrium of Nonconcurrent, Noncoplanar Forces. A force in space requires six factors for its complete specification. These factors may be the three force components along chosen axes and the three moment components with respect to these axes. The force components, of course, may be specified by a magnitude and two directions, and the equivalent of the moment components may be expressed by specifying three coordinates of a point lying on the line of action of the force. Following the rule stated in Art. 28, there are, for a complete system of forces in space, six independent equations, three of resolutions and three of moments. A problem of equilibrium of such a system is statically determinate if it involves just six unknowns. The condition of equilibrium of such a system is given by

$$\begin{aligned} \Sigma \mathbf{X} &= 0 \qquad & \Sigma \mathbf{M}_x &= 0 \\ \Sigma \mathbf{Y} &= 0 \qquad & \Sigma \mathbf{M}_y &= 0 \qquad\qquad \text{Formula VII} \\ \Sigma \mathbf{Z} &= 0 \qquad & \Sigma \mathbf{M}_z &= 0 \end{aligned}$$

As has been pointed out, a moment equation can always be used in lieu of a resolution equation (but not vice versa), so that a problem of equilibrium of forces in space can be solved, if statically determinate, by writing three or more moment equations and a sufficient number of resolution equations to make a total of six (or less if there are deficiencies in the system).

Whenever moment equations are substituted for resolution equations, there is always the theoretical danger of arriving at a set of equations some of which are not independent. The rules on this point are

1. It is not necessary that the axes of moments or of resolution be mutually perpendicular, as might be inferred from Formula VII.

2. Only two resolution axes may lie in the same plane or in parallel planes.

3. Not more than three axes of moment may be concurrent and these three must not all lie in the same plane.

4. Not more than two parallel axes may be coplanar, and if such a pair of coplanar, parallel axes is used, no resolution may be made in a direction perpendicular to the plane containing the two axes.

The danger of writing nonindependent equations has been referred to as "theoretical," for the reason that one skilled in the art of solving problems of statics usually sets up each equation with a view to getting a simple relation involving only a few of the unknowns of the problem, and this procedure greatly minimizes the danger.

85. Force Systems Which Are Not General. Many useful and practical problems of mechanics deal with sets of forces which are not coplanar and yet lack one or more of the elements of a complete, general system of forces in space. For example, there may be some direction in which there are no force components when all forces have been resolved along a particular set of rectangular axes in space. A resolution in this direction for that set of forces is meaningless, and for such a system there are only two independent equations of resolution. If there exists an axis about which no force in the system has any moment, there will be for the system only two independent moment equations. The system of concurrent, noncoplanar forces discussed in the last chapter was a system for which three mutually rectangular axes could be drawn (by placing the origin at the point of concurrency) so that no force produced any moment about any of the three axes. For this system, only the three resolution equations were basically independent. A number of combinations of such deficiencies can exist in space problems, so that such problems may properly have three, four, five, or six unknowns. Fortunately it is not always necessary to recognize the deficiencies before attacking the problem. For example, if it is overlooked that the problem has no components in a given direction and a reaction in that direction is assumed as an unknown, the value found for that unknown will be zero, if the solution is carried out correctly. Some labor can usually be saved if the deficiencies are recognized at the beginning, and a problem which cannot be checked may be found to have an improper number of unknowns.

86. Parallel Forces in Space. Many important practical problems of statics deal with systems of forces all of which are in some one direction, such as the action of gravity on a composite body and the vertical forces used to support the body. Figure 183 indicates such a set of forces and shows a set of rectangular axes with one of the axes, Y, in the direction of the forces. No force in the system has a moment about the Y axis and no force has a component in either the X or Z direction. For this system there are only three independent equations, one of resolution and two of moments. The resolution, in effect, must be in the Y direction since any other direction gives an equation wherein all terms are multiplied by the same cosine. The moment equations may be written with respect to any two axes except an axis in the y direction. If moments are taken with respect to a pair of parallel axes and a resolution equation is also written, the three equations will not be independent. *The resultant must equal the vector sum of all the forces of the system,* in Fig. 183, 120 pounds downward. *The moment of the resultant with respect to any axis must equal the sum of the moments of the forces of the system with respect to that axis.* If the distances in Fig. 183 are in feet, the sum of the moments with respect to the x axis is 300 pound-feet and the sum of the moments with respect to the z axis is 660 pound-feet. The location of the resultant is 2.5 feet from the x axis and 5.5 feet from the z axis. Where there are a number of forces in the system, the calculation should be tabulated as in the following example.

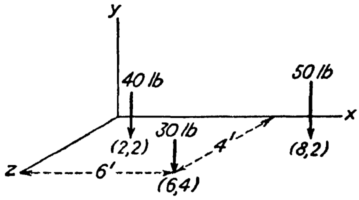


FIG. 183.

Example

Find the magnitude and position of the resultant of the following vertical forces: 24 lb down at $x = 3$ and $z = 5$, 30 lb down at (6,8), 16 lb down at (9,3), 32 lb up at (−3,3), and 12 lb down at (−11,−6).

Since most of the forces are downward, that direction is taken as positive. The moments are calculated with respect to the X and the Z axes, in which $\bar{x}$ and $\bar{z}$ are the coordinates of the resultant force.

Force	x	Px	z	Pz
24	3	72	5	120
30	6	180	8	240
16	9	144	3	48
−32	−3	96	3	−96
12	−11	−132	−6	−72
$R = 50$	. . .	$R\bar{x} = 360$	. . .	$R\bar{z} = 240$
		$\bar{x} =$ 7.2 ft	. .	$\bar{z} =$ 4.8 ft

Problems

1. Find the magnitude and position of the resultant of the following horizontal forces; 20 lb east, 6 ft above the floor and 9 ft south of a vertical wall; 24 lb west, 7 ft above the floor and 4 ft south of the vertical wall; 16 lb east, 9 ft above the floor and 3 ft south of the vertical wall; and 18 lb east, 6 ft above the floor and 5 ft south of the vertical wall.

   Ans. 30 lb east, 6.8 ft above the floor and 7.4 ft south of the vertical wall.
2. Solve Prob. 1, taking moments about a horizontal axis 4 ft above the floor and a vertical axis 5 ft south of the vertical wall.
3. A rectangular door in a horizontal position is 7 ft long from left to right and 4 ft wide. It weighs 40 lb and its center of mass is at the center. The door carries 50 lb at the right corner near the observer, 35 lb at the left corner diagonally across from the 50 lb, 24 lb at the other right corner, and 16 lb at the middle of the left edge. Find the location of a single support which will hold the door in the horizontal position.
4. Find the reactions at the ground in Fig. 176, p. 166.

87. Equilibrium of Parallel Forces. When a system of forces (including the unknowns) is parallel and noncoplanar, it is statically determinate if there are not more than three unknowns, since for such a system there are only three independent equations. If there seem to be less than three unknowns there are one or two unrecognized unknowns or the problem has too many data. When there are too many data the problem is unsolvable or has been partly solved already. The unknowns can be any combinations of magnitudes and positions. All directions are assumed known, a priori, as the direction of parallelism.

Example

The three forces of Fig. 183 are to be put in equilibrium by a vertical force A at the origin, B at a point on the X axis 4 ft from the origin, and C at a point on the Z axis 3 ft from the origin.

Assume A, B, and C all vertically upward.

Moments about the X axis: $40 \times 2 + 50 \times 2 + 30 \times 4 = 3C$; $C = 100$ lb.

Moments about the Z axis: $40 \times 2 + 30 \times 6 + 50 \times 8 = 4B$; $B = 165$ lb.

Moments about an axis parallel to the Z axis and through the line of action of the 50-lb force: $8A = 30 \times 2 + 40 \times 6 - 8C - 4B$; $A = -145$ lb.

Vertical resolution: $40 + 30 + 50 = 100 + 165 - 145$. Check.

Problems

1. A horizontal trapdoor, 8 ft long and 5 ft wide, has two hinges on one 8-ft edge 1 ft from each end. The door weighs 60 lb and has its center of mass at the center. It carries a load of 40 lb 3 ft from one end and 4 ft from the line of the hinges. The door is lifted by a vertical force applied at the corner which is farthest from the load.

Find this vertical force and the reaction of each hinge. Solve by three moments and check by a vertical resolution.

2. Where should the vertical force be applied in Prob. 1 in order that one hinge reaction may be zero? Solve by moments.

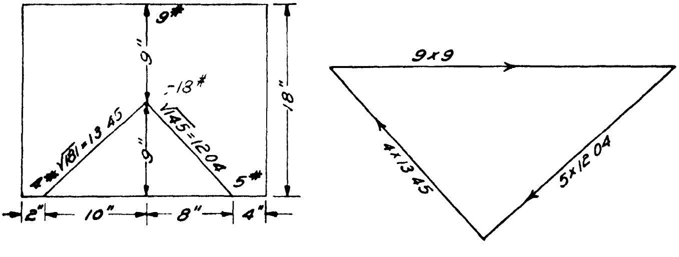
3. A horizontal table, 6 ft long from left to right and 4 ft wide, is supported on three legs. One leg is at each of the left corners and one is at the middle of the right edge. A load of 60 lb is placed 1 ft from the front and 2 ft from the left edge. How much is the load on each leg increased?

4. In Prob. 3, the load is 1 ft from the front edge. How far must it be from the left edge to cause no additional load on one leg? Solve by moments and graphically.

5. A wheelbarrow is 5 ft long from the axis of the axle to points at which the handles are gripped, and is 2 ft wide between the centers of the handles. It carries a load of 160 lb, which is placed 1 ft from the axis of the axle and 4 in. to the right of the middle. Find the reaction of the wheel and of each handle.

6. A horizontal plank, 24 in. long and 18 in. wide, weighs 18 lb and has its center of mass at the center. It is supported at the middle of one 24-in. edge, and 2 in. from one corner and 4 in. from the other corner on the opposite 24-in. edge. Find the reaction of each support. *Ans.* 9 lb; 4 lb; 5 lb.

7. In Prob. 6, consider the downward force of 18 lb as made up of 9 lb, 4 lb, and 5 lb, and consider that these forces form couples with the three reactions. Draw the vector diagram of these couples in the plane of the paper, as in Fig. 184. Show graphically or trigonometrically that this vector triangle closes.



Space Diagram

Vector Diagram of Couples

FIG. 184.

88. Equilibrium of Forces in Space. Figure 185 represents a system of forces and components all of which intersect a line, and this line, OX, has been taken as the X axis. OY and OZ are the two other rectangular axes. The forces have no moment about the X axis and no component in the X direction. There are, for this system, two moment equations and two resolution equations which are independent. Any number of equations can be written, but only four independent ones, so a problem in equilibrium involving a system like Fig. 185 would have four unknowns or it would not be solvable. Problem 11 in the list below is such a problem.

If, to the set of forces in Fig. 185, a force is added at some point such as xy and this force is in the Z direction, there would be moments about all axes but no components in the X direction. There would then be available five independent equations, three resolutions and two moments. Example 1 below is a problem of this type. If the force added at xy had a component in the X direction, the system would be a complete space system with six independent equations available. Example 2 below is such a system.

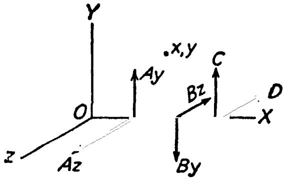


FIG. 185.

Example 1

The mast of a stiffleg derrick (Fig. 186) is 30 ft high. The stiff legs are in vertical planes at right angles to each other. One of these planes extends south and the other extends west of the mast. The stiff legs make angles of 45° with the vertical. The boom is 40 ft long and carries a load of 2000 lb on the end. Find the tension in the stiff legs and the reactions at the bottom of the mast resulting from this load when the boom is elevated 20° above the horizontal in a vertical plane which is north 50° east.

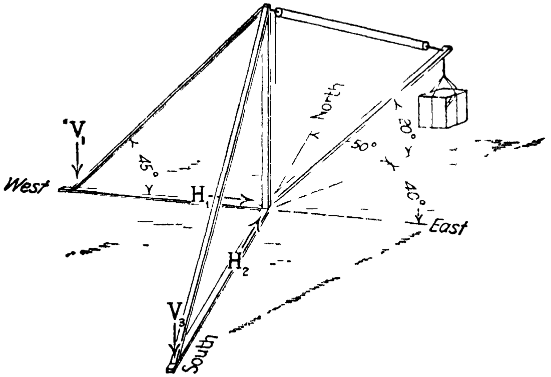


FIG. 186.

Regard the mast, the boom, and the cables which connect them as the free body, acted upon by the load, the tensions in the legs (acting at the top of the mast), the horizontal forces H_1 and H_2 at the bottom of the mast, and the vertical force of the ground V_2 (not shown in the figure) acting at the bottom of the mast. H_1, H_2, and V_2 constitute the three components of a force of unknown magnitude and direction acting at the bottom of the mast. There are five unknowns. The system is deficient by one moment equation since no force has any moment about a particular line in space (the axis of the mast). Begin by taking moments about a horizontal north and south line through the bottom of the mast. This eliminates the reactions at the bottom of the mast and the tension in the stiff leg in the north and south vertical plane. Let **V**$_1$ be the vertical component of the tension in the west stiff leg. Consider this component applied at the base of the leg, where the moment arm of the horizontal component is zero.

$$V_1 \times 30 = 2000 \times 40 \cos 20^\circ \cos 40^\circ = 57{,}585 \text{ lb-ft}$$
$$V_1 = 1919 \text{ lb}$$

The tension in the west stiff leg $= V_1 \sec 45^\circ = 2714$ lb. Take moments about a north and south line through the top of the mast.

$$H_1 \times 30 = 2000 \times 40 \cos 20^\circ \cos 40^\circ = 57{,}585 \text{ lb-ft}$$
$$H_1 = 1919 \text{ lb}$$

Or **H**$_1$ can be found by a resolution east-west.

$$H_1 = 2714 \cos 45^\circ = 1919 \text{ lb}$$

Find **H**$_2$ and the tension in the south leg in a similar way.
For **V**$_2$, resolve vertically.

$$V_2 = 2000 + V_1 + V_3$$

Problems

1. Solve Example 1 when the boom is south 20° east and elevated at 30° with the horizontal.
2. A derrick mast is 40 ft high and weighs 1000 lb. The boom is 50 ft long, weighs 1200 lb, and has its center of mass at the middle. One guy rope runs west and another runs north 15° east. Each rope makes an angle of 30° with the horizontal. The boom is horizontal, south 70° east, and carries a load of 2400 lb. Find the tension in the west guy rope by moments about a line through the lower end of the other guy rope. *Ans.* 4466 lb.
3. In Prob. 2, find the tension in the other guy rope, the vertical reaction at the bottom of the mast, and the direction and magnitude of the horizontal reaction at the bottom of the mast. Find the horizontal reaction by one moment equation and check by resolutions. Check the vertical reaction at the bottom of the mast by moments about the horizontal line through the lower ends of the guy ropes. *Ans.* Vertical reaction at bottom of mast = 7600 lb.
4. At what position of the boom in Prob. 2 will the tension in the west guy rope be the grestest? What is that tension? *Ans.* 4483 lb.
5. A windlass (Fig. 187) is 1 ft in diameter and 4 ft long between bearings. The crank is 2 ft long from the center of the crankpin to the axis of the axle. The force is applied to the crankpin at a distance of 1 ft from the plane through the right bearing perpendicular to the axis. A rope wound round the windlass 20 in. from the left bearing carries a load of 480 lb. The crank makes an angle of 30° to the right of the vertical viewed from the right. The force on the crank is normal to the plane through the axis of the windlass and the crankpin. Find this force and the horizontal and vertical reactions at the bearings. The forces are all in planes perpendicular to the axis so that there are only five unknowns.
Ans. $P = 120$ lb; $V_1 = 265$ lb; $V_2 = 275$ lb; $H_1 = 25.98$ lb; $H_2 = 129.90$ lb.

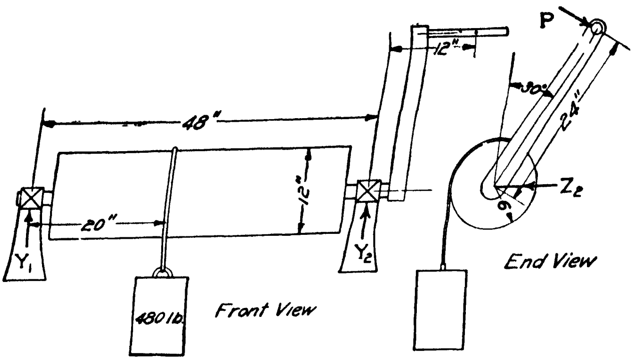


FIG. 187.

6. Solve Prob. 5 when the crank is 45° to the right of the vertical.

7. Solve Prob. 5 when the crank is horizontal toward the right.

In Prob. 7 all the forces are vertical and there are only three unknowns. Problems 5, 6, and 7 serve as a good illustration of the principle that a space problem projected orthographically on three planes can be reduced to three coplanar problems. All geometric distances are foreshortened and appear as the true distances multiplied by the cosine of the angle between the actual line and the projection plane. Forces are indicated by showing the component parallel to to the plane of projection. With P cos 60° placed at the proper position on the crank in the front view of Fig. 187, Y_1 and Y_2 can be solved as a problem of coplanar parallel forces. In the end view, Z_2 should read $H_2 + H_1$, and this view should show $Y_1 + Y_2$ vertically upward at the center of the cylinder. The top view would show only H_1, H_2, and P sin 30°. The projected problems are all coplanar and can be readily solved by graphical methods. The projection method offers no real advantage in any analytical solution.

8. Solve Prob. 5 graphically.

9. In Fig. 185 let $A_y = 30$ lb and $A_z = 40$ lb, 2 ft from 0. Let $B_y = 10$ lb and $B_z = 25$ lb, 4 ft from 0. Let $C = 30$ lb at 6 ft from 0 and $D = 20$ lb at 7 ft from 0. Find the magnitude and direction of the forces required for equilibrium at 0 and at X, 8 ft from 0.

10. A horizontal bar AB is 12 ft long, weighs 252 lb, and has its center of mass 5 ft from the left end A. It is supported by a rope at A and by two ropes at B in a vertical plane perpendicular to the length of the bar. One of these ropes is fastened to a point 4 ft above B and 3 ft in front of the vertical line through it. The other rope through B is fastened to a point 5 ft above B and 12 ft behind the vertical line through it. Find the direction of the rope at A and the tension in each rope. *Ans.* Left, 147 lb; front, 100 lb; rear, 65 lb.

11. The bar of Prob. 10 is supported by a rope at A, a rope at B, and a rope at C, which is 2 ft from B. The rope at B is in a vertical plane perpendicular to the bar and is fastened to a point 4 ft above B and 3 ft in front of the vertical line through it. The rope at C is in a plane perpendicular to the bar and is fastened to a point which is 5 ft above C and 12 ft behind the vertical line through it. Find the four unknowns.

Ans. 100 lb at B; 78 lb at C; 142 lb, vertical component; and 12 lb toward the front horizontal component at A.

SUGGESTION: Begin Prob. 11 by a moment about a vertical axis through A, and a moment about a horizontal axis through A perpendicular to the length of the bar.

12. The bar of Prob. 10 has the end B directly east of A and is supported by a rope at A. A rope at C is fastened as in Prob. 11. A rope at B is fastened at a point 4 ft above B and 3 ft south 60° east of the vertical line through B. Find the five unknowns.

Ans. Tension at B = 113.51 lb; tension at C = 44.27 lb; vertical component at A = 144.16 lb; component west at A = 58.96 lb; component south at A = 6.81 lb.

Example 2

A horizontal trapdoor (Fig. 188) is 8 ft long and 6 ft wide. It is supported by two hinges,[1] each 1 ft from a corner, on an 8-ft edge. It is lifted by a rope attached to the other 8-ft edge at a distance of 2 ft from one corner. This rope passes over a smooth pulley which is 4 ft above the center of the door. The door weighs 60 lb and its center of mass is at the center. Find the tension in the rope and the horizontal and vertical components of the hinge reaction, assuming that the hinges are so constructed that the horizontal force parallel to their line is all taken by the left hinge.

The unknown quantities are the tension in the rope, the horizontal and vertical components of the reaction at the right hinge, two horizontal components and one vertical component at the left hinge. The tension of the rope is made up of three components. Since the direction of the rope is known, the tension can be found when one of these components is determined.

A moment equation about the line of the hinges eliminates five of the unknowns, together with the horizontal components of the tension of the rope. The forces which have moment about this line are the weight of the door and the vertical component of the tension in the rope, with the tension triad placed as shown.

$$60 \times 3 = V_1 \times 6$$
$$V_1 = 30 \text{ lb}$$

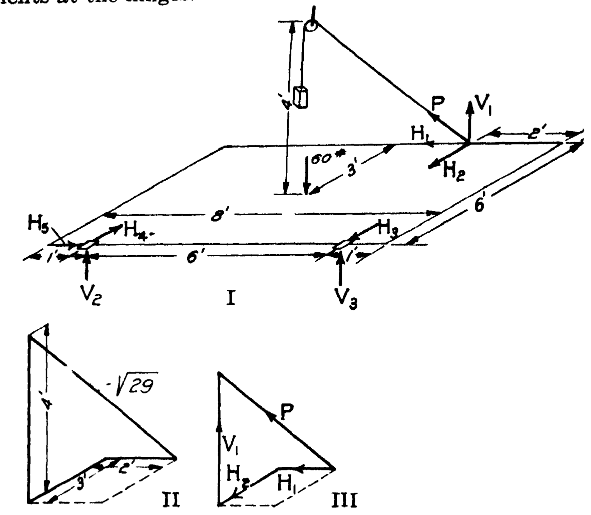
The tension in the rope and its horizontal components may now be computed by resolutions. (These resolutions form no part of the six equations, since the door is not under consideration as the free body.) The space diagram (Fig. 188,II) is similar to the force diagram (Fig. 188,III). All the dimensions of the space diagram are known. The vertical force $\mathbf{V}_1$ of the force diagram is also known. The other forces may be found from the geometry of the similar solids. If $\mathbf{P}$ is the tension in the rope,

$$\frac{P}{30} = \frac{\sqrt{29}}{4}$$
$$P = 40.39 \text{ lb}$$

---

[1] A smooth hinge offers no resistance to rotation about the axis of its pin. Its resistance to small angular movements about axes at right angles to the pin is so small that a hinge is normally assumed to be the equivalent of a ball-and-socket joint. Hinges then exert transverse reactions normal to the pins and longitudinal reactions in the directions of the pins. When two hinge pins are in the same line, the total magnitude of the force along this line may be found, but it is not possible to tell how it is divided between the two hinges except, possibly, by inspection or measurement.

Similarly, $H_1 = 15$ *lb,* $H_2 = 22.5$ *lb.* *These components are convenient in finding* the components at the hinges.



Space Diagram *Force Diagram*

FIG. 188.

Taking moments about an axis which passes through the left hinge and is parallel to the 6-ft edges,

$$V_3 \times 6 = 60 \times 3 - 30 \times 5$$
$$V_3 = 5 \text{ lb}$$

Taking moments about an axis through the right hinge parallel to the 6-ft edges,

$$V_2 \times 6 = 60 \times 3 - 30 \times 1$$
$$V_2 = 25 \text{ lb}$$

Check by a vertical resolution.

$$V_1 + V_2 + V_3 = 30 + 25 + 5 = 60$$

To find $\mathbf{H}_4$ take moments about a vertical axis through the right hinge. This eliminates $\mathbf{H}_3$ and $\mathbf{H}_5$.

$$\begin{aligned} H_4 \times 6 &= H_1 \times 6 + H_2 \times 1 \\ 15 \quad \times 6 &= 90 \\ \underline{22.5 \times 1} &= \underline{22.5} \\ 6H_4 &= 112.5 \\ H_4 &= 18.75 \text{ lb} \end{aligned}$$

To find $\mathbf{H}_3$, take moments about a vertical axis through the left hinge,

$$\begin{aligned} 15 \quad \times 6 &= 90 \text{ counterclockwise} \\ \underline{22.5 \times 5} &= \underline{112.5 \text{ clockwise}} \\ 6H_3 &= 22.5 \text{ clockwise} \\ H_3 &= 3.75 \text{ lb} \end{aligned}$$

The moment of H_3 *about the vertical axis through the left hinge must balance a clockwise moment.* The direction of H_3 must be opposite to the direction of the arrow in Fig. 188. H_3 might have been computed by a horizontal resolution. Resolving as a check,

$$3.75 + 18.75 = 22.5$$

By resolutions parallel to the line of the hinges,

$$H_5 = H_1 = 15 \text{ lb}$$

The problem has been solved by five moment equations and one resolution equation, and partly checked by two resolution equations. Three of the moments were taken about horizontal axes and two were taken about vertical axes.

Problems

13. A horizontal trapdoor, 10 ft long from west to east and 6 ft wide, weighs 60 lb and has its center of gravity at the center. It is hinged on the south edge 1 ft from each end and is supported by a rope attached to the north edge 2 ft from the east end. The rope runs over a pulley 7 ft above a point on the south edge which is 2 ft from the west end. The east hinge takes all the horizontal force in the direction of the hinges. Find the tension in the rope and the components of the hinge reaction.

14. A horizontal trapdoor, 8 ft long from east to west and 5 ft wide, weighs 60 lb and has its center of gravity at the center. It is hinged on the south edge 1 ft from each end. The door is opened until the edge is 4 ft south of the vertical plane through the hinges. It is supported by a bar, one end of which pushes against a stop at the level of the hinges and 10 ft south of a point midway between them. The other end of the bar pushes against the edge of the door 2 ft from the east end.

Find the compression in the bar and the components of the hinge reactions. Solve by means of the moment about the hinges of the vertical component of the force on the bar at the stop.

Ans. Pressure in the bar = 28 lb; vertical reaction of east hinge = 24 lb; vertical reaction of west hinge = 24 lb; horizontal reaction of east hinge toward the south = $25\frac{1}{3}$ lb.

15. In Prob. 14, find the compression in the bar by moments about the hinges of the vertical component and the horizontal component north, applied at the edge of the door.

16. In Prob. 14, imagine the line of the bar to be extended until it intersects the vertical plane through the hinges. Then find the moment of the horizontal component north with respect to the hinge line.

17. A wheelbarrow is 5 ft long from the axis of the axle to the handles and 2 ft wide between the handles. A load of 180 lb is placed 2 in. to the right of the center line and 10 in. from the axis of the axle. The wheelbarrow is held with the axle and the handles in a horizontal plane and is pushed up a 4° incline. Find the reaction at the wheel and the direction and magnitude of the force at each handle, assuming the plane to be smooth.

18. Solve Prob. 17 if a horizontal force of 20 lb, perpendicular to the length, toward the left, is applied 6 in. above the plane of the axle and handles, 2 ft from the vertical plane through the handles. Assume that the left handle takes none of the transverse reaction and assume that the plane is perfectly smooth.

19. The wheelbarrow of Prob. 18 could be pushed up the plane with one hand (providing the operator had sufficient muscular strength). Suppose the operator pushes the wheelbarrow up the plane with his right hand. Find the three components of moment and the three components of force which he must exert. (The moment which twists the handle is customarily called *torque.* The other two are called *bending moments.*)

89. Moments of Space Forces about Axes in Space. Article 78 defined the moment of a force in space about an axis in space and gave an example and some exercises on computing such moments. Article 79 illustrated the use of moments in problems of equilibrium. In all these problems the space situations were such that the geometry of the problems was not particularly formidable. In the more complicated problems of space statics, the procedures of setting up a triad in each case so that one component is parallel to the axis of moments and another (preferably) intersects the axis becomes very laborious. This method was presented and used in the hope that the student might be aided in acquiring a more basic understanding of the meaning of moments. Fortunately, there is a computation procedure which makes the calculation of the moment of any force in space about any line in space just as easy as was the calculation in the simple situations treated so far. The following derivation should be carefully gone through, at least once, so that the student may understand what lies behind the very elegant method of computation which emerges from it. Moment equations play such an important role in space problems that an easy method of computing moments in space becomes a practical necessity.

Figure 189 shows three mutually rectangular axes X, Y, and Z. At point p, whose coordinates are x, y, and z, are shown **X, Y,** and **Z,** the three components of a force passing through p. The axis of moments passes through point q, whose coordinates are a', b', and c', and (to indicate its direction) through another point r, whose coordinates are d, e, and f.

At point q are shown three supplementary axes, X', Y', and Z', parallel to the corresponding axes X, Y, and Z, and on each of these supplementary axes there are shown by double-barbed arrows the moments of the two components of the force triad which have moment about that axis. These are all obtained simply by multiplying each force component by the perpendicular distance from its line of action to the supplementary axis on which the corresponding double-barbed arrow appears. For example, on the Z' axis is the positive moment vector, $\mathbf{X}(b' - y)$, and the negative moment vector, $\mathbf{Y}(a' - x)$. The direction angles of the axis of moments with respect to the coordinate axes are α, β, and γ. To find the moment of the force at p with respect to the axis of moments, the moments shown by double-barbed arrows are resolved in the direction of the axis of moments by multiplying each moment vector by the cosine of the angle which its axis makes with the axis of moments. The following equation results:

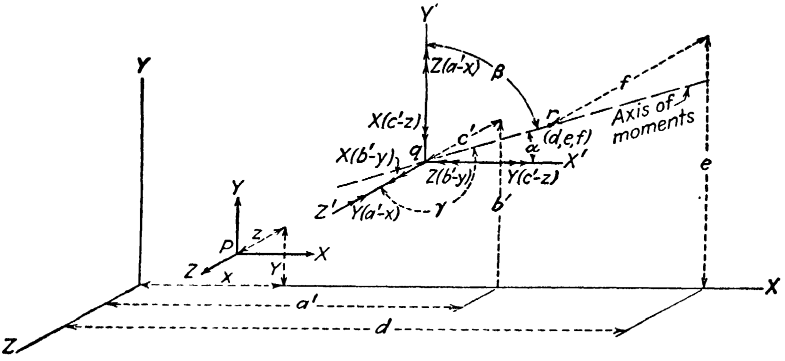


FIG. 189.

$$M = [\mathbf{Y}(c' - z) - \mathbf{Z}(b' - y)] \cos \alpha + [\mathbf{Z}(a' - x) - \mathbf{X}(c' - z)] \cos \beta + [\mathbf{X}(b' - y) - \mathbf{Y}(a' - x)] \cos \gamma \quad (1)$$

In this equation, $(a' - x)$, $(b' - y)$, and $(c' - z)$ are the components of the distance from point p to point q, and in the further discussion these three distances will be called a, b, and c. The distance from p to q will be called the *apparent moment arm* and will usually be designated as abc, its components. If the distance along the axis of moments from q to r is represented by k, the direction cosines of Eq. (1) become $(d - a')/k$, $(e - b')/k$, and $(f - c')/k$. Hereafter, k will be referred to as the *direction distance* and the numbers $(d - a')$, $(e - b')$, and $(f - c')$ will be represented by l, m, and n, and referred to as the *direction numbers*.

When these values are substituted in Eq. (1) and the equation is multiplied through by k, the terms can be collected and written

$$Mk = (Xbn + Ycl + Zam) - (Xcm + Yan + Zbl) \quad (2)$$

Equation (2) is an algebraic form appearing so often in applied mathematical work that it is convenient to write it in the following form, called a determinant:

$$Mk = \begin{vmatrix} X & Y & Z \\ a & b & c \\ l & m & n \end{vmatrix} \qquad \text{Formula VIII}$$

Equation (2) is *exactly* the same as Formula VIII. The formula is simply a more economical way of writing the equation and presents the equation in a form more easily remembered. But it must be clearly understood how the formula is to be read and the reading must come back exactly to what is said in Eq. (2). Start at the upper left corner of the determinant and read diagonally downward to the right to get the product $\mathbf{X}bn$. Begin again at the middle of the top row, read diagonally *downward to the right* and swing under to the lower left corner to get the product $\mathbf{Y}cl$. Then start at the middle of the left column, read diagonally downward to the right and swing up to pick up the upper right corner symbol, making the term $\mathbf{Z}am$. These are the positive terms. The negative terms are obtained by a similar procedure starting at the lower left corner and reading diagonally *upward to the right,* giving the three products $-(lb\mathbf{Z} + a\mathbf{Y}n + mc\mathbf{X})$.*

Example 1

A force in space passes through a point whose rectangular coordinates are (1,3,2). Its components are 160 lb, 220 lb, and 320 lb. Find its moment about an axis passing through the point (4,5,6) and through the point (10,7,9), with distances in feet.

The apparent moment arm components are $(4,5,6) - (1,3,2) = (3,2,4)$.

The direction numbers are $(10,7,9) - (4,5,6) = (6,2,3)$.

The direction distance is $\sqrt{6^2 + 2^2 + 3^2} = 7$.

Then

$$7M = \begin{vmatrix} 160 & 220 & 320 \\ 3 & 2 & 4 \\ 6 & 2 & 3 \end{vmatrix}$$

$$7M = 960 + 5280 + 1920 - 3840 - 1980 - 1280$$

---

* The moment of a force in space about a line in space is the vector product of the force and the distance vector from a point on the line of action of the force to a point on the axis of moments and multiplied, in turn, scalarly, by a unit vector lying in the positive direction of the axis of moments. In vector analysis, this is called a *mixed triple product* and is written $\lambda \cdot (\mathbf{a} \times \mathbf{P})$, in which $\mathbf{P}$ is the force, $\mathbf{a}$ is the apparent moment arm, and λ is the unit vector in the direction of the axis of moments. The unit vector in the direction of the force appears as the set of cosines in Eq. (1). The determinant, Formula VIII, operates to effect this mixed multiplication of (force × arm) · direction.

In tabular form:

$$\begin{array}{rrr}
 & + & - \\
 & 960 & 3840 \\
 & 5280 & 1980 \\
 & 1920 & 1280 \\
\cline{2-3}
 & 8160 & 7100 \\
 & -\ 7100 & \\
\cline{2-2}
 & 7)\underline{1060} & \\
M = & +151 & 43\text{ lb-ft}
\end{array}$$

Figure 190 shows this axis in space with its direction distance, 7 ft, forming the diagonal of a rectangular parallelepiped whose sides are the three direction numbers 6, 2, 3. The *direction of the axis* is the direction in which the direction numbers are read, in this case toward the right, upward, toward the front. The positive moment would be represented by a double-barbed arrow in the direction from q to r. The moment is counterclockwise as viewed from the positive end of the axis. In short, this force passes below this axis.

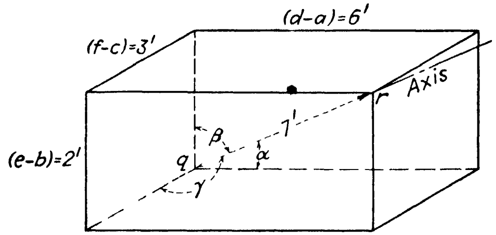


FIG. 190.

This method of computing a moment in space is typical of mathematical operational processes. The data are put into form, then the operation is carried through without any attempt to think out the physical meaning of the steps, and, finally, the result is arrived at without any aid or abetment from common sense. The work should be checked, obviously.

Since two points on the axis of moments are known, a check can always be obtained by repeating the calculation using the second point. The direction numbers for the axis are read in the same order and direction as before unless it is desired to reverse the direction of the axis, in which case the same order and the opposite directions are used. The determinant below is set up for the same force, the new apparent arm, and the same direction numbers as before. Clearly the greatest common factor may be removed from the force components and finally multiplied into the result. This has been done. (The symbol indicating this common factor as a multiplier of the top row is not in common use.) The new apparent arm is (10,7,9) − (1,3,2) = (9,4,7).

$$7M = \begin{vmatrix} (8 & 11 & 16)(20) \\ 9 & 4 & 7 \\ 6 & 2 & 3 \end{vmatrix}$$

$$M = \frac{(846 - 793)(20)}{7} = 151.43 \text{ lb-ft}$$

Keeping the order **XYZ**, *abc*, *lmn* in the rows of this determinant, there are six ways in which the rows can be arranged. Three of them will give the correct result and the other three will give the wrong sign but the proper numerical value. The rows as arranged are force, arm, direction. Of the six ways of arranging the initials of these three words, only one makes an English word, fad, and this one happens to be one of the right ways of arranging the determinant. There are several ways of arranging the order in each row to get the right answer and many ways to get a wrong one. The one chosen here should be easily remembered and it must be strictly followed.

Inherent in the operation of moments by determinants is the necessity for a consistent set of axes and signs. As set up here, X is positive to the right, Y positive upward, and Z positive to the front. The origin may be taken anywhere and one or more negative coordinates may appear in the expression for any one of the three points. Also, any component of the force may be negative. The signs of the values are placed in the determinant with the numbers to which they pertain.

Example 2

Find the moment of the force (−6,4,−2) lb, located at point (3,−5,8), with respect to an axis passing through the origin and through the point (5,7,−1). Distances are in feet.

The apparent arm (to the origin) is (0,0,0) − (3,−5,8) = (−3,5,−8).

The direction numbers are (5,7,−1).

The direction distance is $\sqrt{75}$.

$$M\sqrt{75} = \begin{vmatrix} -6 & 4 & -2 \\ -3 & 5 & -8 \\ 5 & 7 & -1 \end{vmatrix}$$

$$M = \frac{+30 - 160 + 42 + 50 - 336 - 12}{\sqrt{75}} = \frac{-386}{\sqrt{75}} \text{ lb-ft}$$

Problems

1. Check Example 2 by taking the apparent moment arm to the other axis point.
2. A room with a floor 21 ft by 16 ft and a ceiling 12 ft high has a rope attached to one of the lower corners exerting a force of 580 lb toward the opposite upper

corner. Find the moment of this force about an axis extending from the lower corner 16 ft from the point of attachment to the mid-point of the ceiling. (View the room from a 21-ft side, put the point of attachment at the lower left rear corner, and start the axis from the lower left front corner.) Check.

Ans. 2212 lb-ft, clockwise as viewed from the lower end of the axis.

In solving problems of equilibrium by moment equations, when moments are computed by this method the factor k (the direction distance) does not need to be computed since it is the same in all the terms of the equations. In effect, the moment equation is written

$$\Sigma kM_1 + kM_2 + kM_3 \cdots = 0$$

Since the k is never zero, the equation is always valid. For problems in which the geometry is simple, this method of computing moments has no advantages over the simple direct method, but a few such problems need to be solved to give the student skill and confidence in the method.

Example 3

Solve for the force in cord B of Example 2, Art. 77, by taking moments about a line in the ceiling through points A and C. With the axes chosen as parallel to EC for the X axis, vertical for Y, and parallel to AB for Z, the 40-lb force is (0, −40,0) lb, the apparent arm to A is (−3,8, −4), and the direction numbers of the axis from A to C are (9,0,4). Force **B** is described as $(-3\mathbf{B}, 8\mathbf{B}, 3\mathbf{B})/\sqrt{82}$, and with the triad thought of as at point B, the apparent arm is (0,0, −7). (There is only one unknown involved, the magnitude of the force **B**. If all three components of the triad are put in as unknowns, such as $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$, the resulting equation will be correct but will have the appearance of an equation in three unknowns which may be confusing.)

Solution:

$$\begin{vmatrix} 0 & -40 & 0 \\ -3 & 8 & -4 \\ 9 & 0 & 4 \end{vmatrix} + \begin{vmatrix} (-3B & 8B & 3B) \div \sqrt{82} \\ 0 & 0 & -7 \\ 9 & 0 & 4 \end{vmatrix} = 0$$

$$1440 - 480 + \frac{-504B}{\sqrt{82}} = 0$$

$$B = \frac{960\sqrt{82}}{504} = \frac{40\sqrt{82}}{21}\text{ lb}$$

In the examples and problems involving the statement of the positions of points in a formal set of rectangular coordinates, the distances (a,b,c) and (l,m,n) were found by subtracting the coordinates of one point from those of another. As seen from Example 3, it is not necessary to have any formal origin of coordinates. Three mutually perpendicular directions are chosen, as seem most convenient for the problem; every force in the system is thought of as a triad at some convenient point on its line of action (this point may be changed ad lib from equation to equation, but of course in any one equation the triad must be kept all at the same point); and the distances (a,b,c) and (l,m,n) are read from the sketch in the order to the right, upward, to the front.

Problems

3. Solve for the force in cord A of Example 3 by moments about the axis in the ceiling through B and C, and for the force in cord C, using a suitable axis.

4. A rectangular block of stone 8 ft long, 4 ft wide, and 3 ft high is lifted by a force which acts along a line starting on the top of the block 1 ft from an 8-ft side and 2 ft from a 4-ft side toward a point 2 ft above the center of the top of the block. If the block does not slip on the ground, will it turn about a 4-ft edge or about an 8-ft edge? Find the force required to start tipping if the block weighs 18,000 lb.

5. A body of weight **W** is supported at three points. The coordinates of these three points with respect to the center of mass are $A = (1,3,-4)$, $B = (9,-1,2)$, and $C = (-7,-3,10)$. The support at point A is a cord which extends from A to D at $(0,11,0)$. Find the tension in the cord by moments about the axis CB. *Ans.* With apparent arms to B, the determinant equation is

$$\begin{vmatrix} (-1 & 8 & 4)(T/9) \\ 8 & -4 & 6 \\ 16 & 2 & -8 \end{vmatrix} + \begin{vmatrix} 0 & -W & 0 \\ 9 & -1 & 2 \\ 16 & 2 & -8 \end{vmatrix} = 0$$

$$T = \frac{936W}{1580}$$

6. The force at C in Prob. 5 has no component in the Z direction. Using **T** as found in Prob. 5, find the horizontal component of the force at C by moments about a vertical axis through B, and find the vertical component at C by moments about a horizontal axis through B. (These two values can easily be checked by moments without the use of determinants.)

7. Find the three components at B by resolutions.

8. A horizontal trapdoor is 10 ft long and 6 ft wide and weighs 460 lb. It is hinged at the front edge at 1 ft from each end and lifted by a cord attached to the rear edge 2 ft from the right end and to a point 7.5 ft above the left front corner. Take moments about the hinge line (without determinants) and find the vertical component of the tension in the cord. Find the total tension and the other two components. By moments (determinants) about an axis through the left hinge and the point where the cord is attached to the door, find the vertical component at the right hinge. *Ans.* 30 lb.
Find the horizontal component at the right hinge by moments about an axis through the left hinge and the upper point of attachment of the cord.

$$Ans. \rightarrow \begin{vmatrix} 0 & 30 & H \\ 8 & 0 & 0 \\ -1 & 7.5 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -480 & 0 \\ 4 & 0 & -3 \\ -1 & 7.5 & 3 \end{vmatrix} = 0$$

90. Miscellaneous Problems

1. A vertical gate 4 ft wide and 6 ft high is hinged on the west side with two hinges, each 1 ft from a corner. The gate stands in an east-west plane and is held in position by a prop with one end 2 ft above the lower east corner and the other end 4 ft north and 4 ft east of this corner. The gate weighs 120 lb and a wind from the south exerts a uniform pressure of 6 lb per sq ft on the gate. Find the force in the prop and the components of the hinge reaction, assuming the bottom hinge to carry all the vertical load.

2. A solid rectangular block 6 by 4 by 2 ft, weighing 5600 lb, is placed with the 4- by 6-ft faces horizontal and hinged to a support on the lower front 6-ft edge at points 1 ft from the corners. It is held by a cable attached to the upper right rear corner and passing through a point 6 ft above the center of the top face of the block. Solve for the six unknowns. *Ans.* $T = 600$ lb.
3. A bar AB is 15 ft long, weighs 60 lb, and has its center of gravity 8 ft from A. The end A rests on a smooth horizontal floor at the intersection of two vertical walls, one of which runs east and the other south from the corner. The end B is elevated 30° above the horizontal and is held by a horizontal rope from B to a point directly over A. The vertical plane through the bar and the rope makes an angle of 40° with the plane which runs east. Find the four unknown forces. *Ans.* 60 lb, 55.42 lb; 35.63 lb; 42.46 lb.

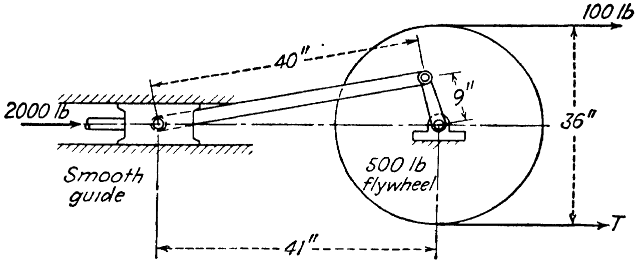


FIG. 191.

4. A bar 29 ft long, weighing 84 lb, with its center of gravity at the middle, stands with the lower end on a smooth floor at the corner and has the 30° inclination as in the preceding problem. The upper end supports a load of 252 lb and is held by a rope 16 ft long running perpendicular to the east wall and a rope 12 ft long running perpendicular to the south wall. Find the five unknowns.
   Ans. Tension in 16-ft rope = 224 lb; tension in 12-ft rope = 168 lb; vertical reaction at floor = 336 lb.
5. In Prob. 4, find the three couples. Resolve vectors which represent these couples along the three coordinate axes, and show that their vector sum is zero.
6. A slender rod 10 ft long has its mass center 7 ft from the left end B and is held in a horizontal position by three cords. One cord at B is attached to a point C 4 ft above, 4 ft to the left, and 2 ft back of B; another cord at B is attached to a point D 9 ft toward the front from C. The third cord is attached at A. Find the forces in the three cords and locate the point in the ceiling 4 ft above the bar at which the cord from A is attached.
7. Figure 191 shows the crosshead, connecting rod, crank, and flywheel of a simple side-crank engine. The main bearings are 24 in. apart and the flywheel is midway between them. The center of the crankpin is 8 in. outside the right main bearing. At the position shown the tension in the top belt is 100 lb and the piston-rod force is 2000 lb. Assuming no acceleration, find the components of the bearing reactions.

8. A uniform rod 12 ft long hangs horizontally 12 ft below a horizontal ceiling. A cord at the left end of the rod is attached to a point in the ceiling 5 ft forward from the point directly over the left end of the rod. A second cord is attached to the rod 2 ft from the left end and extends to a point in the ceiling 9 ft back of and 6 ft to the left of the point directly over the left end. A third cord is attached at the right end. Locate the point in the ceiling to which the third cord extends and find the forces in all cords in terms of the weight of the rod.

9. Find the torque and bending moment at a point midway between the flywheel and the right bearing in the crankshaft of Prob. 7.

10. A uniform rectangular block 3 ft north-south, 4 ft east-west, and 6 ft high is supported at the lower southeast and southwest corners and held by a cable attached 2 ft west of the upper southeast corner and running to a point 3 ft south of the lower southeast corner.

a. Assume that the block weighs 1200 lb and that all the east-west component is carried by one of the support points, and find the six unknowns.

b. Apportion the east-west component between the two support points so that the total horizontal force at each support bears the same ratio to the vertical component at that support as does the horizontal to the vertical at the other support. Find the minimum coefficient of friction required at the supports (same at both) if friction is depended on to hold the block in place.

Ans. $\mu = 0.206$.

11. Solve Prob. 2 if the cable is attached 3 ft west of the upper southeast corner and 2 ft south of the lower southeast corner. *Ans.* $T = 1050$ lb.

12. A horizontal rectangular plate is 4 ft long from east to west and 3 ft wide from north to south. It is supported by three cords attached to a horizontal ceiling 8 ft above. One cord is fastened to A, which is 1 ft from the northwest corner on the west edge. The second cord is fastened at B, which is on the north edge 1 ft west of the northeast corner. This cord is 17 ft long from B to the ceiling. The third cord is fastened at the southeast corner C. This cord is fastened to the ceiling 1 ft east and 4 ft south of the vertical line through C. Find the six unknowns when the load is 480 lb at the center of the plate.

Ans.	C	B	A
Vertical	168	96	216
East	21	116.81	−137.81
South	84	−136.94	52.94

Of the nine results given in the answer, six depend upon the equilibrium equations. Three of these are at A, two at B, and one at C. The remaining terms, together with the angles which are not given, may be found from these six, and the geometrical data of the problem.

13. The plate of Prob. 12 is supported at C as in that problem. The cord at B is attached to the ceiling 9 ft east and 12 ft north of the vertical line through B. The direction of the cord at A is not specified. Where must the center of gravity be located?

Ans. On a line through A intersecting the east edge at $^{14}\!/_{15}$ ft north of C.

14. In Prob. 13, the load is 480 lb midway between the east and west edges. Find the components of the tension in the rope at A.

Ans. Vertical, 217.6 lb; west, 122.4 lb; south, 48 lb.

15. A uniform cylinder 10 ft long and 2 ft in diameter fitted at the ends with flanges 4 ft in diameter is rolled along uniformly by a cable unwinding from the top of the cylinder (Fig. 192). If it is being rolled up a plane inclined 3 ft vertically on 4 ft horizontally, with the cord parallel to the plane and the axis of the cylinder horizontal, how far can the cable tangent point be from the middle of the cylinder before slipping impends? Derive a general expression in μ and get a numerical value when $\mu = 0.4$. *Ans.* 3 ft, when $\mu = 0.4$.

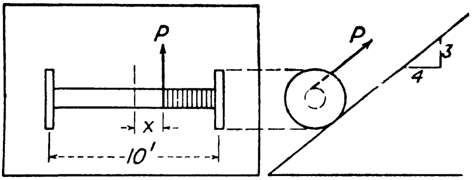


FIG. 192.

16. At point H on the space framework of Fig. 176 put the three following components: **V** vertically downward, **H** in the direction FE, and **P** in the direction GH. Pass a sectioning plane between planes DEF and ABC. This plane cuts six two-force members, and the forces in these members are the unknowns in the problem of equilibrium of the portion of the framework above the sectioning plane. Using DB as an axis of moments, set up a determinate and find the force in CF.

$$\textit{Ans.}\quad \begin{vmatrix} T & -V & H \\ -16 & -24 & 4\ 5 \\ -9\cos 30^\circ & 12 & -4\ 5 \end{vmatrix} + \begin{vmatrix} 0 & CF & 0 \\ 0 & 0 & 10 \\ -9\cos 30^\circ & 12 & -4\ 5 \end{vmatrix} = 0$$

17. Take moments about the line DE to find the force in BF, and about the line DF to find the force in EB of the frame of Prob. 16.

18. Complete the solution of the problem for the forces in DC, DB, and DA.

CHAPTER 10

MACHINES AND FRICTION

91. Simple Machines. In the science of mechanics, a machine is defined as a contrivance for overcoming a force at one point by a force applied at another point. The six simple machines, formerly called the *mechanical powers* and now sometimes called the *classical machines*, are the *lever*, the *pulley*, the *wheel and axle*, the *inclined plane*, the *wedge*, and the *screw*. The first three named operate on the principle of the lever and the latter three on the principle of the inclined plane, and these two principles can be easily shown to be special cases of the principle of virtual work. The force applied to the machine with a component in the direction of the displacement is called the *effort* and the force which the machine overcomes is called the *resistance*. The effort does positive work on the machine, and the machine does positive work on the resistance. In the writing of equations, it is properly considered that the resistance does negative work on the machine. The points at which the effort and resistance act are called the *driving point* and the *working point*, respectively. The ratio of the displacement of the driving point in a given time to the displacement of the working point in the same time is called the *velocity ratio*. When the velocity ratio is a variable the displacements on which its definition is based must be infinitesimal. The ratio of the effort to the resistance is called the *mechanical advantage* of the machine. A serious and common error is to assume that the velocity ratio is the same as the mechanical advantage. This would be true only in an ideal machine with *no lost work*. It is very far from true in many actual machines.

In all machines a certain amount of the work done by the effort is dissipated to heat by friction of various kinds and is called *lost work*. The work done by the effort is called the *total work* and that part done on the resistance is called the *useful work*. The ratio of the total work to the useful work is called the *efficiency*. When the machine is run backward so that the effort becomes the resistance and vice versa, the ratio of the work input to the work output is called the *reversed efficiency*.

92. The Inclined Plane, the Wedge, and the Screw. Figure 193 shows a body being moved up an inclined plane by a force parallel to the plane. If the plane is perfectly smooth, the work done by the force P in moving the body a distance l is the work required to lift the body a vertical distance h. The negative work of gravity on the body of W pounds mass is Wh, and $Pl - Wh = 0$, or $P = W(h/l)$. The velocity ratio is l/h, and this is also the mechanical advantage (the plane being smooth).

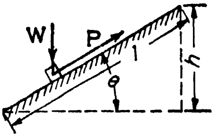


FIG. 193.

There is always some friction between the body and the inclined plane. In Fig. 193 the normal pressure between the body and the plane is $W \cos \theta$, and if the coefficient of friction is μ, the lost work is $W\mu l \cos \theta$, and $Pl = Wh + W\mu l \cos \theta$, whence

$$W = \frac{P(l/h)}{1 + \mu \cot \theta} \tag{1}$$

In Eq. (1), the factor $1/(1 + \mu \cot \theta)$ is the efficiency, and this factor multiplied by the velocity ratio is the mechanical advantage.

If the machine is run in reverse, W is the effort and P is the resistance. The friction force is reversed with $Wh = Pl + W\mu l \cos \theta$, whence

$$P = W\frac{h}{l}(1 - \mu \cot \theta) \tag{2}$$

In Eq. (2), the factor $(1 - \mu \cot \theta)$ is the reversed efficiency. The student can easily show that the reversed efficiency is less than the efficiency in this case. This is always true in a machine which has a velocity ratio greater than unity. If the reversed efficiency is zero, the machine will not "overhaul" or run backward. If $(1 - \mu \cot \theta) = 0$, $\mu = \tan \theta$ and the plane is inclined at the angle of friction. In this case, the efficiency factor in Eq. (1) is the reciprocal of $(1 + \tan \theta \cot \theta)$, which is 0.5. This is the basis of the rule that *the efficiency of a machine must be* 50 *per cent or less if it is to be self-locking.*

This self-locking or self-supporting property of a machine is very convenient in many cases such as lifting tackle, screw jacks, differential hoists, clamps, etc. *Efficiencies less than* 50 *per cent do not guarantee self-locking.*

Figure 194 shows a body being moved up an inclined plane by a horizontal force. The velocity ratio in this case is b/h, and for the ideal frictionless case, $W = Pb/h$. But with a coefficient of friction of μ, the friction force is $\mu(W \cos \theta + P \sin \theta)$, and the efficiency becomes

$\dfrac{1 - \mu \tan \theta}{1 + \mu \cot \theta}$, *which, for any values of* μ *and* θ, *is less than for the case where the effort was parallel to the plane.* The reversed efficiency is again zero when $\mu = \tan \theta$, but the efficiency at that value of μ is less than 50 per cent. With values of μ greater than $\tan \theta$, the reversed efficiency becomes negative. *A machine is self-locking if the reversed efficiency is zero or negative.*

The wedge is two inclined planes placed together and pushed between two bodies to force them apart. If one of the two bodies is stationary, the other becomes the resistance. The velocity ratio is the ratio of the length of the wedge to the increase of thickness in that length. Its efficiency is less than that of an inclined plane with the same coefficient of friction because there are frictional resistances at both faces.

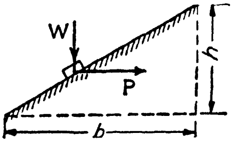


FIG. 194.

The screw with square threads has the same action as the plane of Fig. 194. The plane is wound into a helix and constitutes the thread of the screw. The nut can be considered as producing the resistance W. The force P represents the force at the end of the wrench multiplied by the ratio of the length of the wrench to the radius of the thread. Screw jacks usually have square threads and should have the same efficiency as the inclined plane of Fig. 194, but the bearing cap at the top of the screw must turn relative to the screw, making the condition really more like that of the wedge. The friction at the cap is usually about the same as that at the thread so the efficiency is about half that of the inclined plane. Threads cut with a slope, such as V threads, have a sidewise wedging action, which makes their efficiency less than that of square-cut threads.

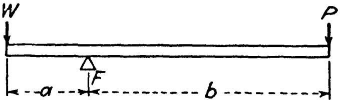


FIG. 195.

93. The Lever, the Pulley, and the Wheel and Axle. The lever is, mechanically, simply a beam with loads at three points. One point remains stationary and is called the *fulcrum.* The effort is applied at another point, and the resistance at the third. Figure 195 shows a "lever of the first class," with the effort at P, the fulcrum at F, and the resistance at W. With this arrangement the velocity ratio is b/a. Figure 196 shows a bent lever, sometimes called a *bell crank.* The classes of levers, depending on the arrangement of the fulcrum, effort, and resistance, are now considered somewhat quaint and are seldom used in speech or in writing. The "law of the lever" is the same thing as the principle of moments. In the early literature of mechanics there were a myriad of "laws," *one, in fact, for nearly every conceivable device. These laws, fortunately and properly, are tending to be forgotten.* Such of them as were not replaceable by the principles of moments and resolutions or by the principle of conservation of energy were either useless or erroneous or both. Falling into disuse, also, is much of the terminology of simple machines, except "mechanical advantage," which seems doomed forever to be misused in place of "velocity ratio."

Figure 197 shows a simple pulley. It is equivalent to a lever with the fulcrum at the middle. A cord passing over the pulley is subjected to the effort on one side and the resistance on the other. The velocity ratio is obviously unity. For the ideal pulley, free of all friction, the mechanical advantage is also unity.

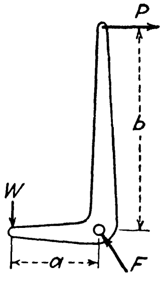


FIG. 196.

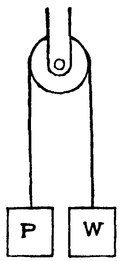


FIG. 197.

In Fig. 197 let the radius of the pulley to the center of the cord be R and the radius of the hole through which the supporting pin passes be a. Then if the pulley turns through an angle θ (radians), the mass P descends a distance $R\theta$, the mass W rises $R\theta$, and the bearing slides on the pin $a\theta$. Neglecting the weight of the pulley, the normal force of the pin in the bearing is *approximately* $W + P$ and the friction force there is $\mu(W + P)$. Equating the work done by the weight of P to the sum of the useful work and the lost work, and canceling θ,

$$PR = WR + \mu(W + P)a \tag{1}$$

Solving for W,

$$P\frac{1 - \mu a/R}{1 + \mu a/R} = W \tag{2}$$

Since the velocity ratio is unity, the coefficient of P in Eq. (2) is the efficiency of the pulley.

In this discussion of the efficiency of the pulley, the energy lost in bending and unbending the rope was neglected. If the rope is replaced by a thin metal band which suffers no permanent deformation as it is bent and unbent to the radius of the pulley, this loss is very small, but with a large stiff rope on a small pulley it may amount to more than the friction loss at the bearing. Wire ropes are lubricated to reduce wear between the strands and to increase the efficiency by lessening the rope loss. It was emphasized above that the normal force used in the equations is approximate. The approximation was not entirely in neglecting the weight of the pulley and rope, and this will be explained later in an article on bearing friction. When the coefficient of friction is large and the ratio of the two radii is small, the approximation may be considerably in error.

Pulleys are grouped together into sets, with some of the pulleys fixed and the others moving with the load or resistance. Figure 198 shows such a set with one fixed and one moving pulley. The velocity ratio for this arrangement is 2, and the effort is moving in the opposite direction from the resistance. When the effort and resistances have opposite directions of movement, the velocity ratio obtainable is equal to the number of pulleys used, provided the ropes are parallel. One pulley can always be saved by letting the effort move in the same direction as the resistance. In multiple pulley sets the force in each of the intermediate ropes is the effort on the pulley it leaves and the resistance on the next one, but the efficiency of the whole device is not the product of the efficiencies of the pulleys. The mechanical advantage is, as before, the product of the velocity ratio and the over-all efficiency. The reversed efficiency is not the product of the reversed efficiencies of the several pulleys.

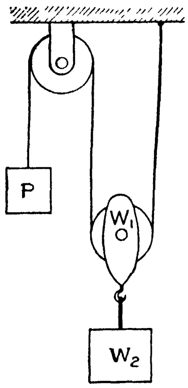


FIG. 198.

Figure 199 shows an arrangement by which the velocity ratio of a group of pulleys can be made 2^n, where n is the number of pulleys available. If e is the efficiency of a single fixed pulley as shown in Fig. 197, the efficiency of the same pulley moving as in Fig. 199 is $(e + 1)/2$ and the efficiency of the whole rig of Fig. 199 with n pulleys is $[(e + 1)/2]^n$. The reversed efficiency, of course, is smaller but the device is obviously not self-locking. The mechanical advantage is the efficiency multiplied by the velocity ratio, as is always the case.

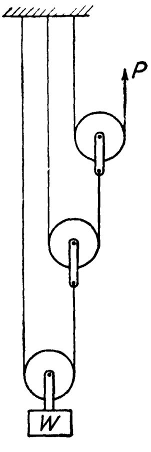


FIG. 199.

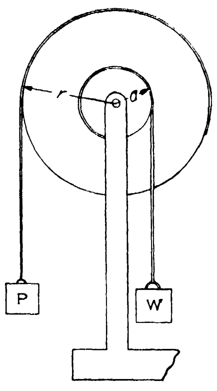


FIG. 200.

Figure 200 shows a device known as the *wheel and axle* or *Chinese windlass*. The effort P unwinds rope from the large wheel while the load is lifted by the rope winding onto the smaller wheel, called the *axle* in the old name for the device. The actual axle may be smaller still, as shown in the figure. The velocity ratio is r/a.

Problems

1. Using R for the larger radius, r for the smaller, and a for the radius of the actual axle, find the efficiency and the reversed efficiency of the wheel and axle.

   Ans. Efficiency $= \dfrac{1 - \mu a/R}{1 + \mu a/r}$. Reversed efficiency is the same with R and r interchanged.

2. In terms of e, the efficiency of one fixed pulley of Fig. 197, and n, the total number of pulleys used, find the efficiency of the arrangement of Fig. 198, and find the increase in the mechanical advantage when another pair of pulleys is added.

   Ans. Eff. $= \dfrac{(e + e^2 + e^3 + \cdots + e^n)}{n}$; added advantage $= e^{n+1} + e^{n+2}$.

3. Figure 201 shows a differential hoist. As P descends, rope winds off the small drum and onto the intermediate one. If the three radii are 20 in., 6 in., and 5 in., what is the velocity ratio?

4. If the bearings in Fig. 201 are 1 in. in diameter, for what coefficient of friction is the device self-locking, and what are the efficiency and the mechanical advantage with this coefficient of friction? (Take other data as in Prob. 3).

5. The pitch at A of the differential screw of Fig. 202 is three threads to the inch and that at B is four threads to the inch. The screw is turned by means of a rod which is 20 in. long from the axis of the screw to the point of application of the force. Find the velocity ratio between the motion of the point of application of the force and of nut B. *Ans.* 1508.

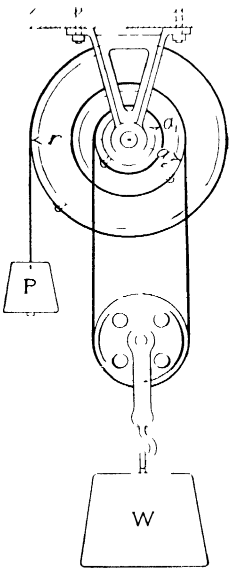


FIG. 201.

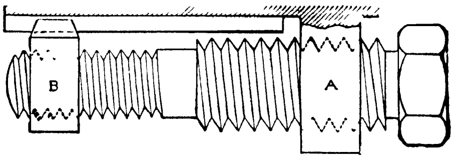


FIG. 202.

6. If the threads of the screw of Prob. 5 are square cut and lubricated so that the coefficient of friction is 0.05, what is the efficiency and the mechanical advantage? Assume the pitch diameter at A as 1.5 in. and at B as 1 in.

7. In tests reported in *American Machinist* a differential screw had 3 threads per inch on the larger screw and 3.5 threads per inch on the smaller. With good lubrication a force of 27,000 lb was produced by a force of 95 lb at the end of a lever arm of 47.75 in. Find the efficiency and the mechanical advantage.

   Ans. Efficiency = 4.5 per cent; velocity ratio = 6300; mechanical advantage = 284.

8. With the same lever arm as in Prob. 7 but with the larger screw only in action, a force of 82 lb at the end of the lever arm produced a force of 27,000 lb. Find the efficiency, the velocity ratio, and the mechanical advantage. (The efficiency was so greatly improved that the mechanical advantage was actually higher in spite of the great decrease in velocity ratio.)
9. If the efficiency of the larger screw in Prob. 7 was the same as that found for the same screw in Prob. 8, what was the reversed efficiency of the smaller screw in Prob. 7?

94. Friction. Without friction, the world as we know it would be impossible. No vehicle could roll on a road or on rails and no one could walk on solid ground. Perhaps there would be no solid ground, as the soil would flow down to the sea like water and the beaches would flow back into the waters that cast them up. Without fluid friction, which seems at first sight such a great nuisance in hindering the motion of a ship or an airplane, neither the ships nor the planes could develop the vortexes around their propeller blades which enable them to start in the first place. In engineering there are just two basic problems connected with friction, how to eliminate or reduce it in cases where it is not needed and how to increase it or make it more effective in those cases where it is advantageous. The friction laws are stated in Art. 35 and have been used in a large number of problems and examples. The following articles apply the same simple laws and the law of rolling friction to some problems in which the friction element is of greatest importance.

95. Bearing Friction. In the discussion of the efficiency of pulleys in Art. 93, the normal force on the bearing was taken as equal to the vector sum of the forces on the pulley, as a good approximation. If the forces on the pulley are in equilibrium, the resultant of the friction force and the normal force at the bearing is the equal and opposite of the resultant of all other forces acting on the pulley.

Figure 203 shows a shaft of radius a turning in a very loose bearing in a clockwise direction. No such bearing would be used in practice. It is drawn in this way in Fig. 203 in order to show details more clearly. When the shaft is turned it climbs up on the bearing until it reaches a point at which it slides down as fast as it rolls up. At this point, the resultant pressure makes an angle with the normal which is equal to the angle of friction for the surfaces of the shaft and the bearing. The normal force makes the same angle with the radius to the point of contact. If a circle is drawn tangent to the line of action of N and concentric with the shaft, its radius is $a \sin \phi$. This circle is called the *friction circle*, and it is the key to graphical solutions of problems involving bearing friction. It can also be used to advantage in many analytical solutions.

If the wheel and shaft are running as an idler with uniform velocity and P and Q are belt forces, the resultant of the moments of P and Q with respect to the center of the shaft must be equal to the moment of the friction force about that center.

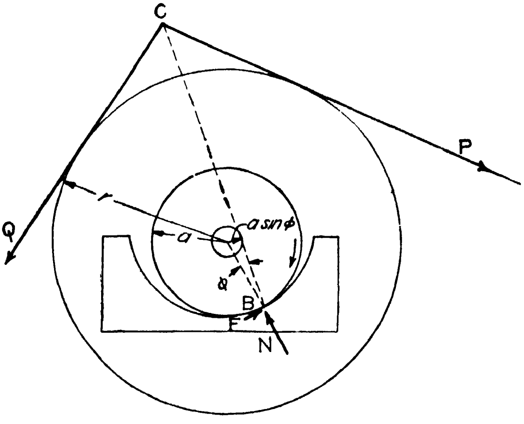


FIG. 203.

Example

A 200-lb wheel with a 2-ft diameter has an axle 6 in. in diameter resting in bearings on either side of the wheel. A rope passing over the wheel hangs vertically downward on both sides. A force of 240 lb on one side is required to lift a 200-lb load on the other. Find the coefficient of friction (Fig. 204).

Taking moments about the axis of the shaft,

$$3F = 240 \times 12 - 200 \times 12 = (240 - 200) \times 12 = 480 \text{ lb-ft}$$
$$F = 160 \text{ lb}$$

Since all the applied forces are parallel, their resultant is equal to their sum. The resultant is 640 lb vertically downward.

$$\sin \phi = \frac{F}{R} = \frac{160}{640} = 0.25$$
$$\phi = 14°29'$$
$$\mu = \tan \phi = 0.258$$

The student will understand that this bearing is not lubricated. The coefficient of friction for a well-lubricated bearing should be many times smaller than 0.258.

Since all the applied forces are vertical, this problem may be conveniently solved by moments about the point of contact of the shaft and bearing. The equation of moments about this point B of Fig. 204 is

$$200(12 + 3 \sin \phi) + 200 \times 3 \sin \phi = 240(12 - 3 \sin \phi)$$

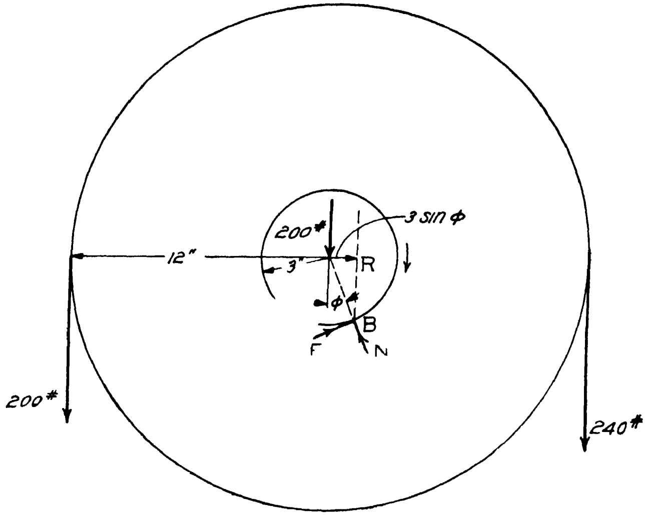


FIG. 204.

Problems

1. In the example above, let the force Q be vertical and the force P be horizontal toward the right. Find the coefficient of friction and the position of the point of contact of the shaft and bearing.
   Ans. $R = 467$ lb; $\mu = 0.365$. The normal at the point of contact makes an angle of $20°04' + 30°58' = 51°02'$ with the vertical.
2. An axle is 4 in. in diameter and the wheel is 2 ft in diameter. The wheel and axle together weigh 200 lb. The coefficient of friction is 0.16. A flexible rope passes over the wheel and carries a load of 300 lb on one end. What load on the other end will just lift the 300 lb? *Ans.* 321.6 lb.
3. Solve Prob. 2 if the axle is 2 in. in diameter instead of 4 in.

Approximate solutions of problems of bearing friction are often sufficiently accurate. One approximation is the assumption that the resultant is equal to the normal pressure. This is equivalent to $R = N$, and $\sin \phi = \tan \phi$. When the coefficient of friction is small this method involves little relative error. Another approximation is the calculation of the resultant (but not the moment) on the assumption that $P = Q$. In Prob. 2, if it is assumed that $P = Q$ in the calculation of the resultant,

$$R = 300 + 300 + 200 = 800 \text{ lb, approximately}$$

If it is assumed that $R = N$,

$$F = 0.16 \times 800 = 128 \text{ lb}$$

Taking moments about the axis of the shaft,

$$\begin{aligned}(Q - 300) \times 12 &= 128 \times 2\\ Q - 300 &= 21.3 \text{ lb}\\ Q &= 321.3 \text{ lb}\end{aligned}$$

which differs from the answer of Prob. 2 by only 0.3 pound, or 1 part in 1000. It is not probable that the coefficient of friction is known with anything like this degree of accuracy.

Problems

4. A wheel 6 ft in diameter has an axle 4 in. in diameter. The wheel and axle together weigh 500 lb. The coefficient of friction is 0.12. One end of a rope which passes over the wheel lifts a load of 400 lb. Find the tension on the other end of the rope required to lift this load when applied vertically downward. *Ans.* $Q = 408.67$ lb.

5. Solve Prob. 4 if the tension is applied horizontally. *Ans.* $Q = 406.57$ lb.

6. Derive an expression for the efficiency of the pulley of Fig. 197, neglecting the weight of the pulley and the energy required to bend the rope, as before, but using the correct value for N instead of the approximate value.

Ans. $\dfrac{1 - (\mu a/R\sqrt{1 + \mu^2})}{1 + (\mu a/R\sqrt{1 + \mu^2})}$.

7. Derive an exact expression for the efficiency of one of the pulleys of Fig. 199.

Ans. $1 - \dfrac{a\mu}{R\sqrt{1 + \mu^2}}$.

8. Figure 205 shows an eccentric with strap and connecting rod actuating a crosshead for a valve mechanism. Assume that the eccentricity is 1 in. and that the connecting rod is horizontal at its highest position. Neglect the weight of the eccentric, strap, and connecting rod. If the radius of the eccentric is 3 in. and the length l is 24 in, what is the direction of the force which the rod exerts on the crosshead pin (a) with good lubrication giving a coefficient of friction of 0.04, and (b) with no lubrication and a coefficient of friction of 0.4? Assume rotation clockwise.

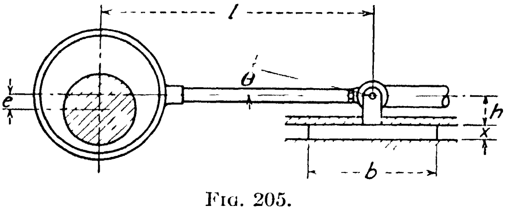


FIG. 205.

9. Solve Prob. 8 for the lowest position of the eccentric.

10. If the horizontal component of the force on the crosshead pin in Prob. 8 is 60 lb, what is the torque on the eccentric shaft resulting from the eccentric strap forces?

96. Rolling Friction. When a cylinder rolls on a surface, the surface is depressed somewhat by the pressure which the cylinder exerts on it. This is shown on an exaggerated scale in Fig. 206. If the material of the surface is resilient, it will spring back so as to push against the cylinder on both sides of the lowest point. If its resilience is small, it will come back slowly so that practically all the pressure will be in front of the lowest point of the cylinder. In any case, the reaction of that part of the surface in front of the cylinder will be greater than the reaction on the other side, and the resultant of all the reactions will be somewhat as shown in the figures. The cylinder may be regarded as continuously climbing a small hill. Experiments are said to show that the distance of the resultant reaction from the line of the load **W** is practically constant for a given material and is independent of the diameter of the cylinder. This distance, f_r of Fig. 206, is the *coefficient of rolling friction.* For steel on steel the coefficient of rolling friction is from 0.02 to 0.03 inch.

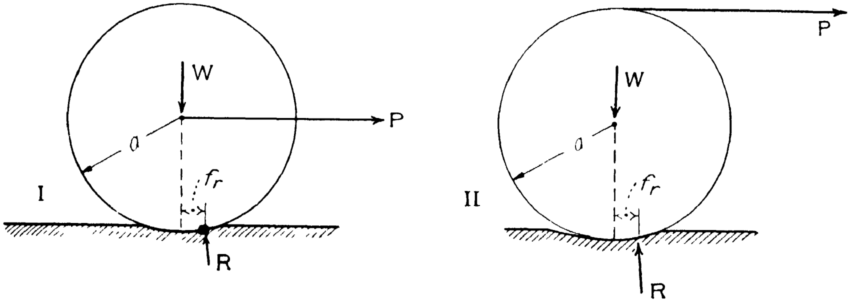


Fig. 206.

In Fig. 206,I, the cylinder is rolled along a horizontal surface by a horizontal force applied at the center. If moments are taken about the point at which **R** intersects the periphery, the moment arm of the force **P** is practically equal to the radius of the cylinder. The moment arm of the weight **W** is the coefficient of rolling friction f_r. The moment equation is then

$$Wf_r = Pa \tag{1}$$

In Fig. 206,II, the cylinder is rolled along by means of a force parallel to the surface applied at the top. The moment arm of this force is the diameter of the cylinder.

$$Wf_r = 2Pa \tag{2}$$

Problems

1. Figure 207 represents a cable reel with spool diameter 1.5 ft and flange diameter 3 ft. The flanges are rolling along horizontal ground where the coefficient of rolling friction is 3 in. Find the horizontal force on the cable required to move the reel. *Ans.* $W/9$.

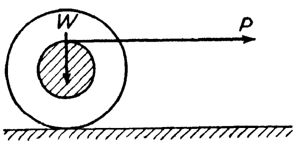


FIG. 207.

2. Solve Prob. 1 if the cable is pulled off horizontally from the bottom of the drum.
3. Solve Prob. 1 if the reel, with the cable coming off the top horizontally, is rolled up a 30° inclined plane with the same coefficient of rolling friction.
4. Find the force in Prob. 1 if the reel is to be started over an actual obstruction, like a square block, 4 in. high. (In problems of rolling friction it is customary to neglect the vertical small distance from the bottom of the wheel to the point through which the resultant passes. In the case of an actual obstruction of considerable height, this is not usually done. Take moments about the corner of the block.)
5. Figure 208 represents a mass of W lb on cylindrical rollers. The coefficient of rolling friction is 0.04 in. at the top and 0.03 in. at the bottom. If the cylinders are 10 in. in diameter and weigh 100 lb each, and the load W is 2400 lb, find the force P to keep the load moving on a horizontal track. *Ans.* 17.4 lb.

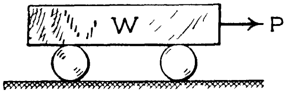


FIG. 208.

6. The wheels and axles of a mine car with live axles weigh 360 lb and the car with load weighs 3960 lb (including wheels and axles). If the axles are 3 in. in diameter and run in bearings where the coefficient of friction is 0.07 and the wheels are 18 in. in diameter and run on rails where the coefficient of rolling friction is 0.05 in., what horizontal force is required to keep the car going after it is started? *Ans.* 64 lb.
7. An automobile weighing 3364 lb and having wheels 30 in. in diameter runs uniformly up a slope of 41 ft on 840 ft horizontally. The load normal to the road is distributed equally on all four wheels, the wind resistance is 180 lb, and the coefficient of rolling friction on all wheels is 0.5 in. Find the friction force parallel to the road acting on the rear wheels. (Neglect bearing friction of front wheels.) *Ans.* 300 lb.

97. Torque, Angular Displacement, and Work. If the pulley of Fig. 197 is turning with uniform speed and P is descending, the cord supporting the mass P has a tension equal to the weight of P since P is in equilibrium moving in a straight line with uniform velocity. Gravity does positive work on P and the cord does an equal amount of negative work on P in any given time. Similarly the cord on the other side does positive work on W while gravity does an equal amount of negative work. If the pulley is considered as the free body, the cord on the left side does positive work and the cord on the right side does negative work. These two quantities of work are not equal, the difference being made up by the negative work which the friction force at the pin does on the pulley. If the pulley turns through an angle θ the work done by the force in the left cord is $PR\theta$, and

$$PR\theta = WR\theta + \mu(W + P)a\theta,$$

which is the same as Eq. (1) of Art. 93 before the θ was canceled. But PR is the moment about the center of the shaft of the force P in the left cord, WR is the moment of the force in the other cord, and $\mu(W + P)a$ is the moment of the friction force about the same center. In each term, the moment times the angle (in radians) of turning is seen to be the work. Moment turning or tending to turn a rotative device like a shaft, wheel, pulley, crank, rotor, or the like is called *torque.* Torque multiplied by the angular displacement about an axis normal to the plane of the torque equals work. If the torque is in pounds-feet and the angle is in radians, the work is in foot-pounds. This relation is easily understood from the action of turning a crank with the force constant and always at right angles to the crank arm. Figure 209 shows a force P turning a crank of length r. If the crank is turned through an angle θ, the force acts through a distance $r\theta$ and does work equal to $Pr\theta$, or the torque Pr multiplied by θ. Letting U symbolize work, T torque, and θ the angle, the relation above is usually written

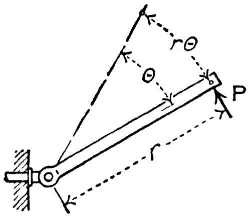


FIG. 209.

$$U = T\theta \tag{1}$$

The work per revolution is

$$U = 2\pi T \tag{2}$$

If the torque is variable,

$$U = \int T\,d\theta \tag{3}$$

98. Belt Friction. Figure 210,I represents a pulley and a belt. The belt is in contact with the pulley through an angle of α radians. The tension of the belt at the left point of tangency with the pulley is P_1. The tension at the right point of tangency is P_2. The tension P_2 is greater than P_1 so that the belt tends to turn the pulley in a clockwise direction.

$$P_2 - P_1 = \text{friction between belt and pulley}$$

Figure 210,II shows a small element of the length of the belt. This element is enclosed between radii which make an angle $d\theta$ with each other. The tension on the left end of the element is P; on the right end is $P + dP$. The difference between these two tensions is dP. This difference is equal to the friction between the belt and the pulley. Since the two tensions, P and $P + dP$, are each tangent to the pulley, they make an angle $d\theta$ with each other, and each tension makes an angle $d\theta/2$ with the tangent at the middle of the element. The normal reaction of the pulley on the element of the belt is obtained by a resolution along the radius BO. The component of P is $P \sin d\theta/2$, which,

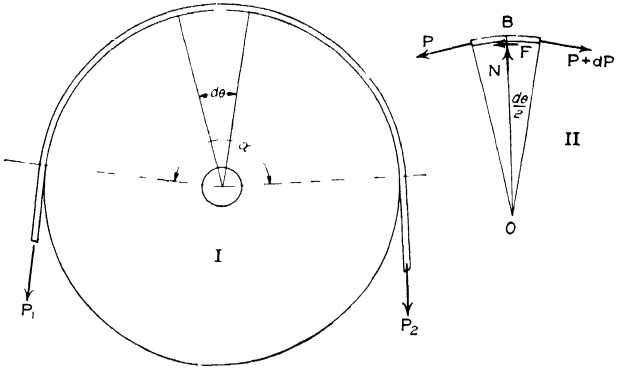


FIG 210.

for a small angle, is equal to $P\,d\theta/2$. When the differentials of the second order are omitted, the component of $P + dP$ is also $P\,d\theta/2$. The total normal force on the element is

$$N = P\,d\theta \tag{1}$$

If motion is impending between the belt and the pulley the friction F of the small element of the belt is

$$F = \mu N = \mu P\,d\theta = dP \tag{2}$$

Separation of the variables of Eq. (2) gives

$$\frac{dP}{P} = \mu\,d\theta \tag{3}$$

$$\log_e P = \mu\theta + C \tag{4}$$

When $\theta = 0$ at the left point of contact, $P = P_1$ which, substituted in Eq. (4), gives $C = \log_e P_1$. When this value of C is substituted in Eq. (4) the result is

$$\log_e P = \mu\theta + \log_e P_1 \tag{5}$$

At the right point of contact, where $P = P_2$ and $\theta = \alpha$,

$$\log_e P_2 = \mu\alpha + \log_e P_1 \tag{6}$$

$$\mu\alpha = \log_e \frac{P_2}{P_1} \tag{7}$$

$$e^{\mu\alpha} = \frac{P_2}{P_1} \tag{8}$$

Example

A belt is in contact with a pulley through an angle of 180°. The tension at one point of tangency is 100 lb and the coefficient of friction is 0.4. What is the maximum tension at the other point of tangency?

$$\log_e \frac{P_2}{P_1} = 3.1416 \times 0.4 = 1.2566$$

If a suitable table of Napierian logarithms were available, the ratio of P_2 to P_1 could be found by looking up the number whose Napierian logarithm is 1.2566. Without such a table, the Napierian logarithm must be reduced to the common logarithm.

$$\log_{10} \frac{P_2}{P_1} = 0.5457$$

$$\frac{P_2}{P_1} = 3.51$$

$$P_2 = 100 \times 3.51 = 351 \text{ lb}$$

The tension of 351 lb is the greatest possible without slipping. The actual tension may be very much less than 351 lb.

It is not necessary that the pulley of Fig. 210 be stationary. It may be turning in either direction at any speed. The speed may even be changing, but that will not be considered here. In this figure, P_2 is the larger tension and its torque about the center is greater than that of P_1. If the pulley is turning clockwise, it is a *driven pulley* and is taking work from the belt. This work is transmitted through the shaft and is taken out by bearing friction and by other things such as another pulley, which is a *driving pulley* delivering work to another belt. If the pulley of Fig. 210 is rotating counterclockwise, it is a driving pulley. The relations of the belt tensions and the friction forces between the belt and the pulley are just the same whether the pulley is driving or driven. If the pulley is rotating rapidly, there is an acceleration effect which decreases the normal pressure between the belt and the pulley and so diminishes the maximum difference between P_2 and P_1, but this will not be considered here.

Problems

1. Solve the above example if the belt is in contact through only 90°. (This can be solved without tables.) *Ans.* 187.3 lb.

2. If the pulley of the example is running at 360 rpm and its radius is 1.5 ft, what horsepower is it transmitting? (1 hp = 550 ft-lb per sec)
3. A fixed cylinder 4 ft in diameter has its axis horizontal. A rope hangs vertically from one side of the cylinder and supports a mass of 1200 lb. The coefficient of friction between rope and cylinder is 0.4. If the rope is in contact with the cylinder through an arc of 3 ft, what is the minimum tension at the other point of tangency? *Ans.* $P_1 = 658.6$ lb.
4. If there is a vertical pull of 100 lb at one point of tangency of the cylinder of Prob. 3 and the rope is in contact through an arc of 2 ft, what is the maximum pull at the other point of tangency? *Ans.* $P_2 = 149.2$ lb.
5. Solve Prob. 4 if the contact is through an arc of 4 ft; an arc of 8 ft; and an arc of 12 ft. *Ans.* $P_2 = 222.6$ lb; 495.3 lb; 1102.3 lb.
6. With the answer given for Prob. 4, how could the answers for Prob. 5 be found without the use of logarithms?
7. Two fixed cylinders of the same diameter have their axes parallel, horizontal, and at the same height. A cord passing over the cylinders hangs vertically on both sides and supports a mass of 100 lb on one side. If the coefficient of friction between the cord and the cylinders is 0.2, what is the largest mass which can be hung from the other side without causing the cord to slip on the cylinders? *Ans.* 188 lb.
8. Show that it makes no difference in Prob. 7 whether the cylinders are of the same diameter or have their axes at the same height.
9. As a belt goes around a pulley it is continually stretched as the tension increases, and the belt "creeps" on the pulley. If every foot of the belt of Prob. 2 going on at the slack side appears on the tight side as 1.001 ft, how much power is being lost due to this effect? *Ans.* 0.1 per cent.
10. Figure 211 represents a slab of material lying on a slope. If the two bodies are of different materials or suffer different average temperature changes as diurnal or other heating and cooling takes place, the upper one will creep downhill along the lower one. The figure shows the friction force condition during contraction of the slab. If the length of the slab is 10 ft and the coefficient of friction is 0.6, locate the neutral point N, assuming uniform pressure under the slab. Find the motion of each of the two ends during a total contraction of 0.2 in., relative to the plane, if $\alpha = 30°$.

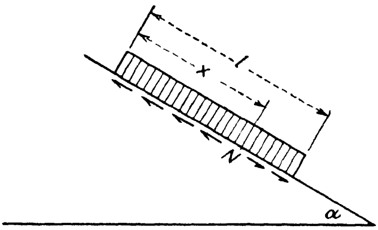


FIG. 211.

11. Find the total creep of the slab of Prob. 10 during a cycle of expansion and contraction of 0.2 in.
12. Solve Prob. 11 if the slab is assumed to be supported at the ends only.
13. If the slab of Prob. 10 were supported at the ends (equally) and at the middle, what proportion of the load would have to be carried at the mid-point to prevent creep?
14. Find x of Fig. 211 as a general expression in terms of μ, l, and α.
15. Derive a general expression for the creep for a complete cycle in terms of the factors of Prob. 14 and δ, the expansion.

99. Friction Brakes. An ordinary friction brake is a device by which a shoe is pressed against a rotating wheel. The force tangential to the wheel is the normal force multiplied by the coefficient of friction, and the normal force sometimes needs to be very great to effect the retarding force required. This difficulty has been overcome since ancient times by band brakes arranged to be "self-actuating." Figure 212 shows a simple (possibly the original) arrangement of the device. With the rotation in the direction indicated by the curved arrow, the brake is effective. With the opposite direction of rotation, the effectiveness is much reduced. The lever and pins can be so arranged that the device is effective for both directions of rotation.

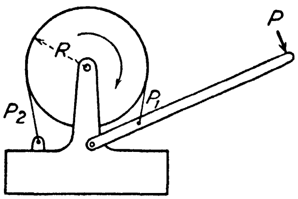


FIG. 212.

Problems

1. Find the braking torque per pound of force perpendicular to the lever in Fig. 212 if the lever is 4 ft long with the band attached 4 in. from the lower end, assuming the band to be in contact through 200° and to make an angle of 60° with the lever at the attached end and with the coefficient of friction 0.4.
2. Solve Prob. 1 with the direction of rotation reversed.

Internal self-energizing brakes consists of rigid shoes mounted inside the brake drums and so hinged that the friction forces have moments about the hinges tending to turn the shoes toward the friction surfaces. They are much more difficult to analyze than are band brakes and they require a complicated mechanism to make them effective for both directions of rotation.

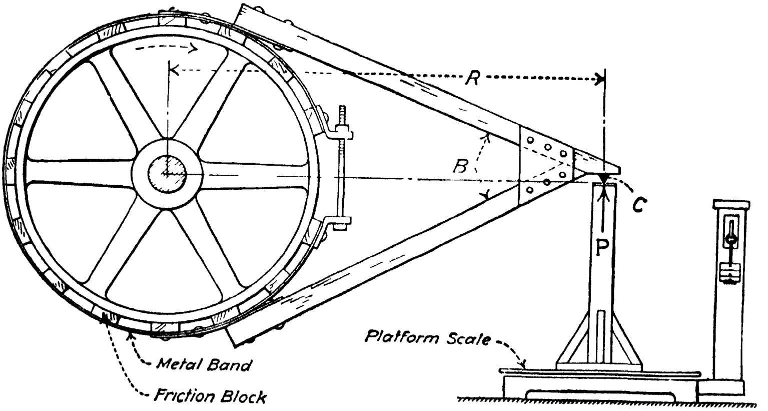


FIG. 213.

100. Absorption Dynamometer. Any device by which the rate of transfer of energy can be measured is called a *dynamometer.* Devices by which energy is absorbed and dissipated in the process of measuring are called *absorption dynamometers.* Figure 213 shows an absorption dynamometer making use of a friction brake. The form shown consists of a number of wooden friction blocks placed around the rim of the pulley. The blocks are fastened to a flexible metal band. The ends of the band are connected together by bolts. By turning the nuts on these bolts the tension may be varied and the pressure of the blocks on the pulley may be adjusted to give any desired value to the friction. The arm B ends with a knife-edge C at a distance R from the axis of the pulley. The knife-edge rests on a vertical support, which stands on a platform scale. When the pulley is turning in the direction of the arrow, the friction of the blocks transmits a torque to the brake. This torque is balanced by the upward force P from the support to the knife-edge.

Suppose that the pulley is held stationary and that a force P is applied to the knife-edge at right angles to the radius R. If the torque of this force is sufficient to balance the torque of friction, the knife-edge will describe a circle of radius R about the axis of the pulley. The circumference of this circle is $2\pi R$, and the work done by the force P during one revolution is

$$U = 2\pi RP = 2\pi T \qquad (1)$$

in which T is the torque.

If the brake is held stationary by the reaction of the support and the pulley is rotated, the work per revolution is the same as it would be if the pulley were stationary and the brake were rotated. In all cases, the work per revolution is $2\pi T$.

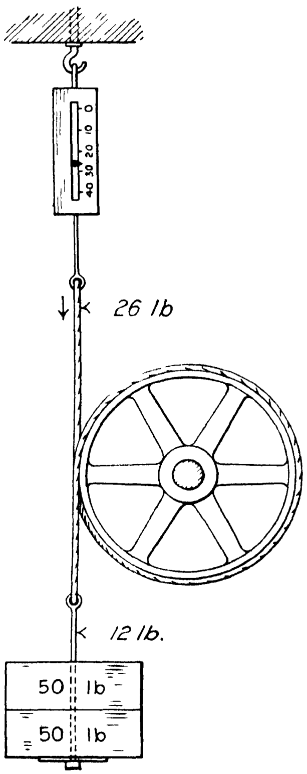


FIG. 214.

Problems

1. The flywheel of an engine makes 300 rpm. A friction brake on the wheel has a moment arm of 90 in. Find the horsepower when the reaction at the end of the brake arm is 420 lb. *Ans.* 180 hp.
2. The arm of a brake which is attached to the pulley of a motor is 30 in. long. The motor makes 1200 rpm. Find the horsepower when the reaction at the support is 60 lb. If the current used to drive the motor is 130 amp at 220 volts, what is the efficiency? *Ans* 34.27 hp; 89.3 per cent.
3. The pulley of Fig. 214 is 10 in. in diameter and the rope is $\frac{3}{4}$ in. in diameter.

The total weight is 112 lb and the spring balance reads 26 lb. Find the horsepower when the pulley is making 800 rpm. *Ans.* 5.86 hp.

4. What must be the coefficient of friction between the rope and the pulley of Prob. 3?
5. The flywheel of Prob. 1 is cooled by water which is held against the rim of the wheel by centrifugal force. The water is supplied to the wheel at 60°F and is evaporated at 210°F. How many pounds of water are required per hour?

101. Cradle Dynamometer. The output of a motor may be measured by means of a cradle dynamometer. The motor is mounted on a frame which is supported by a pair of knife-edges. These knife-edges are in line with the axis of the shaft of the motor. The torque of the belt on the motor tends to rotate the frame about the knife-edges. Since the knife-edges are in line with the axis of the pulley, the direct pull of the belt has no effect on the equilibrium. The torque of the belt is balanced by the weights P and the poise Q on a horizontal arm attached to the frame. When the motor is revolving clockwise with the belts extending toward the left, as shown in Fig. 215, the tension is greater in the upper belt and the torque from the belt to the armature is counterclockwise. This torque is transmitted electromagnetically from the armature to the field and then transmitted mechanically from the field to the dynamometer.

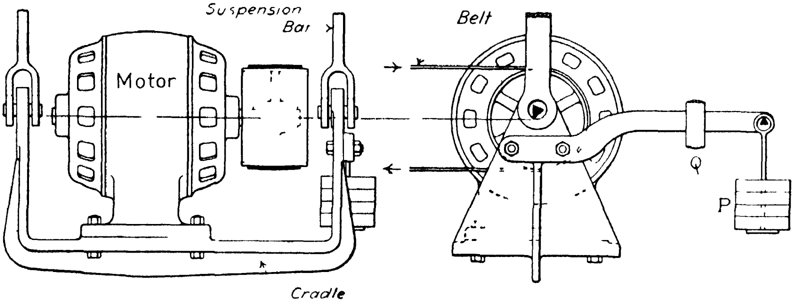


FIG. 215.

When a dynamo which is driven by a belt is to be tested, either the rotation must be opposite that shown in Fig. 215 or the arm which carries the balancing weights must extend toward the left.

The calculation of power of a cradle dynamometer is the same as the calculation for a friction brake. The work per revolution is $2\pi T$.

Aircraft engines are sometimes mounted as cradle dynamometers with an instrument on which the operator can read the torque at any time. Combined with the tachometer reading, the torque meter enables the operator to keep a continuous check on the power output of the engine.

102. Transmission Dynamometer. A transmission dynamometer transmits and measures power without dissipating it by transfer into heat. Figure 216 shows one type of a dynamometer driven by a belt. The pulley *B* is driven by pulley *A* through the dynamometer pulleys *C* and *D*. The dynamometer is arranged to measure the pull *Q* in the part of the belt running from *C* to *B* and the pull *P* in the part of the belt running from *B* to *D*. The difference $P - Q$ multiplied by the radius of the pulley *B* (measured to the center of the belt) gives the torque. Each dynamometer pulley is mounted on a frame which is supported by knife-edges. The knife-edges which support pulley *C* are in the plane of the center of the belt running from *A* to *C* and the knife-edges which support pulley *D* are in the plane of the center of the belt running from *D* to *A*. The tension in these parts of the belt, therefore, exerts no torque on the frames which support the dynamometer pulleys. The moment on the left frame is the product of the pull *Q* multiplied by the diameter of pulley *C* (from center to center of belt), and the moment on the right frame is the product of the pull *P* multiplied by the diameter of pulley *D*.

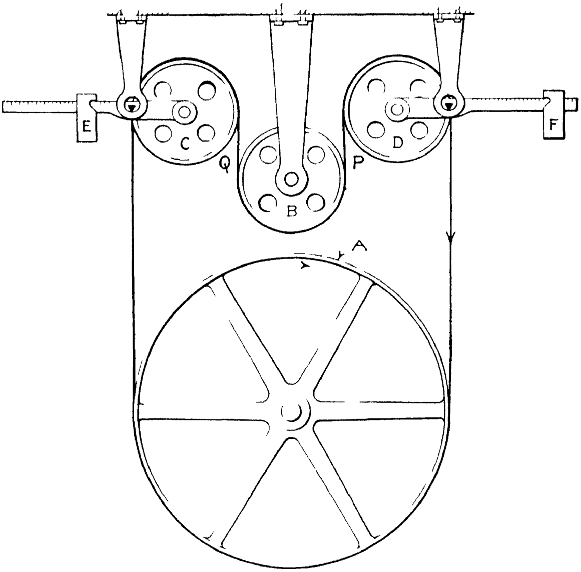


FIG. 216

Figure 217 shows both frames attached to a single lever, at equal distances on opposite sides of the fulcrum *G*. The moment on this lever measures the difference $Q - P$. It is not absolutely necessary that the belt should run vertically from A to C and from D to A. The belt may run at any angle, provided the plane of the center passes through the line of the knife-edges on each side.

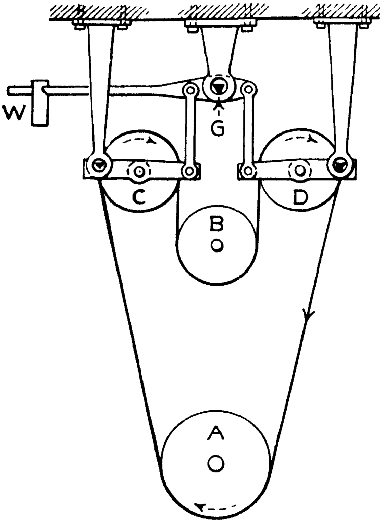


FIG. 217.

Modern instrumentation makes it possible to measure the elastic deformation of a rotating shaft accurately enough to determine the amount of torque transmitted. This development tends to render the classical dynamometers obsolete.

103. Friction Locking. In Fig. 218, the resultant of the applied forces on the small body, including its own weight, is represented by Q, which makes an angle θ with the normal to the surface CD. Figure 218,II is the force diagram when θ is equal to the angle of friction ϕ. The force diagram closes. If ϕ is the angle of sliding friction, the body moves with constant speed after it has been started by a small additional force. If ϕ is the angle of starting friction, the body is in the condition of incipient sliding. If θ is less than the angle of starting friction, the body cannot be moved no matter how great the force **Q** and is in a condition known as *friction locked.*

If θ is greater than the angle of friction, as in Fig. 218,III, the force polygon does not close, and the body moves with increasing speed in the direction of CD.

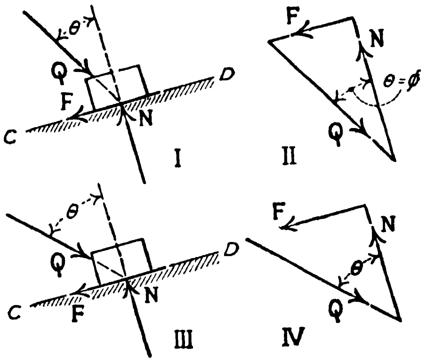


FIG. 218.

If a line which makes an angle ϕ with the normal to a surface is rotated about the normal, a conical surface is generated, as in Fig. 219. If the resultant of the forces applied at the vertex falls within this cone, the angle with the normal is smaller than the angle of friction, and the forces cannot produce motion parallel to the surface, no matter how great they may be. This is called the *cone of friction.* When friction is depended on to prevent movement, the resultant force on the friction surface must be kept definitely inside the cone of friction. When motion along a surface is necessary in a functional machine, it must be so designed that the resultant force on the sliding surface cannot get within the cone of friction. Some of the most serious mistakes in machine design have resulted from failure to follow these simple rules. In this relation, common sense is likely to prove an unreliable guide. Every case deserves a free-body diagram and a careful analysis.

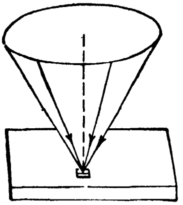


FIG. 219.

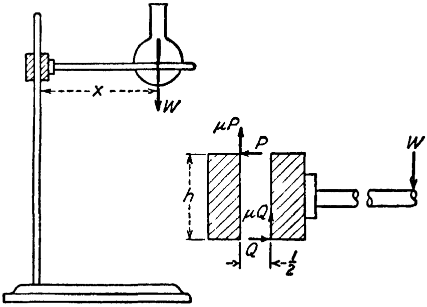


FIG. 220.

Figure 220 shows a familiar device known as a *chemical ring stand.* The sleeve fits closely but not tightly on the upright rod. If the center of gravity of the ring and load is far enough out, the device will stand wherever it is put without clamping.

Example

If the sleeve of a ring stand is h in. high and runs on a vertical rod $\frac{1}{2}$ in. in diameter, how far out must the center of gravity be to ensure against slipping down? The sketch at the right is the detail of the sleeve showing the forces **P** and **Q** which the rod exerts normally inside the sleeve and the upward friction forces at the pressure points. The points of contact are assumed at the extreme top and bottom points of the sleeve since this is the most unfavorable assumption. The free body is the sleeve, the ring, and the load.

Resolving horizontally,

$$P = Q \text{ (a couple)} \qquad (1)$$

Resolving vertically,

$$W = 2P\mu \qquad (2)$$

Since the two friction forces are equal, their resultant is up the center of the rod. Measuring x from the center and equating the two couples,

$$Ph = 2P\mu x \qquad (3)$$

from which $x = h/2\mu$. The diameter of the rod is immaterial.

Problems

1. A wooden desk drawer 24 in. wide and 30 in. deep (horizontally) has handles 4 in. from the sides. A cord is tied to one of the handles and the drawer is pulled open by the cord which is kept horizontal and perpendicular to the front of the desk. How far out can the drawer be pulled if the coefficient of friction is 0.4? *Ans.* 22 in.
2. A freight elevator 8 by 6 ft. in plan and 6 ft high runs in angle guides at the corners and is hoisted by a cable at the middle. Assuming that the guides are neglected and rusty with a coefficient of friction of 0.5, find how far from the center of the cage the resultant of the weight of the cage and its load may be without causing the cage to bind while being lowered.
Ans. 6 ft: Since this is outside the cage there is no danger.
3. Figure 221 shows a V plunger used to reverse the motion of a grinding or milling machine. When the dog attached to the carriage engages the slope on the plunger, it forces it down and actuates the reversing mechanism. Neglect the weight of the plunger and the spring force holding the plunger and find the minimum value for x in terms of d, h, μ, and α. Assume the same coefficient at the top and the sides of the plunger.

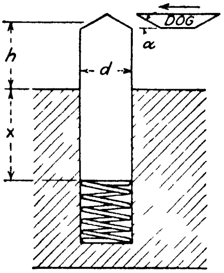


FIG. 221.

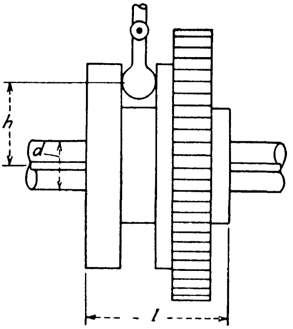


FIG. 222.

4. Figure 222 shows a gear being shifted along a splined shaft by a ball-point lever in a groove. If the lever exerts a force parallel to the shaft, under what conditions will locking occur?
5. An eccentric crosshead as shown in Fig. 205 locked and broke when b was 4 in., t was 2 in., and h was 4 in. Assume the force at the pin as horizontal and find the coefficient of friction. *Ans.* $\mu = 0.4$.
6. If the elevator of Prob. 2 carries a load of 4 tons (including its own weight) and the center of gravity of cage and load is 2 ft off-center toward one corner, what force is required to hoist the cage?
7. Figure 223 shows a pair of grips used for holding a specimen in a testing machine. The sloping backs of the grips are lubricated and have a coefficient of friction of 0.1. The faces of the grips are artificially roughened and have a coefficient of friction of 0.5. What should be the slope angle of the grips to ensure holding?
8. Figure 224 shows a testing machine grip for testing fabric, rubber, etc. The specimen is passed over the small cylinder and pinched at p against the vertical

bar. The cylinder is rigidly fastened to the arm of length a and this arm is hinged to the bar. Assume p a point at the end of the horizontal diameter of the cylinder, take $a = \frac{5}{8}$ in., $b = \frac{1}{4}$ in., $d = \frac{3}{8}$ in., and $t = \frac{1}{32}$ in. and find the coefficient of friction required between the fabric and the grip (assumed the same at all surfaces) to support any load **T**. Would the coefficient need to be greater, or less, with thicker material?

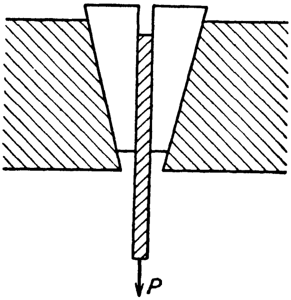


FIG. 223.

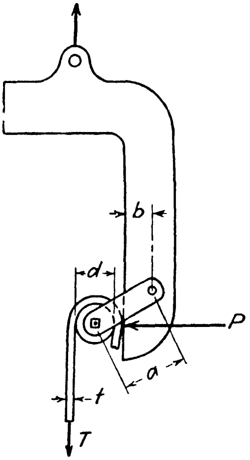


FIG. 224.

9. A ring stand like that of Fig. 220 has $h = 2$ in., $x = 7.75$ in., and diameter of vertical rod $= 0.5$ in. The friction coefficient between rod and sleeve is 0.25. If the sleeve is turned one complete revolution by a small horizontal force acting on the end of the arm, how far will the sleeve descend along the rod during the motion? *Ans.* 0.577π in.

CHAPTER 11

FLEXIBLE CABLES

104. Cables with Uniform Horizontal Loads. A graphical method of determining the positions of and forces in light cords was presented in Chap. 6. When the load on the cable is distributed or consists of a great many small concentrated loads, the graphical method becomes very laborious and for many such cases analytical methods are easier and more elegant. An especially easy case is that of a cable carrying a load which is uniform per horizontal unit. The hangers, floor system, and stiffening truss of a suspension bridge are so much greater than the weight of the supporting cables that the weight of the combined structure may be very nearly uniform per unit of horizontal distance.

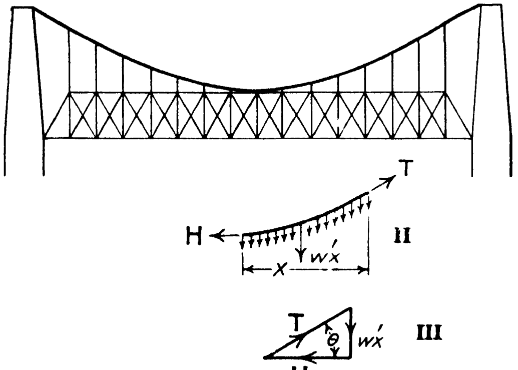


FIG. 225.

Figure 225 shows such a structure. (The cables extending over the towers, across other spans, perhaps, and finally to anchors are not shown.) Figure 225,II is the free-body diagram for a part of the cable extending from the lowest point to any point a horizontal distance x to the right of the lowest point. H represents the total force in the cable at the lowest point and T is the tension at the point x. Let w' represent the load per unit length on the cable and $w'x$ the total load on the part taken as a free body. Since the load is uniformly distributed horizontally, the resultant of all the loads falls at the mid-point of the distance x. Since the cable is flexible, *the force in the cable at every point is in the same direction as the cable at that point.* Figure 225,III shows the force triangle for the free body. The problem of the force in the cable at any point and of the direction of the cable at that point can easily be solved by moments and resolutions. Take as a free body a portion of the cable, beginning at the lowest point and extending to any point a distance x horizontally and a distance y vertically above the lowest point. The force of gravity on the free body is $w'x$ located at $x/2$ from the origin. Let H be the tension at the origin (the lowest point) and T be the tension at the point x, y. Taking moments about point x, y,

$$Hy = \frac{w'x^2}{2} \tag{1}$$

$$y = \frac{x^2}{2(H/w')} \tag{2}$$

Equation (2) is the equation of a parabola. The quantity H/w' is a length, the length of load whose weight equals the tension at the lowest point. The slope at x, y is $w'x/H$.

By resolutions or from the force triangle,

$$T = \sqrt{H^2 + (w'x)^2}$$

Concentrated loads or changes from point to point of the uniform load offer no special difficulties so long as the loading remains symmetrical so that the horizontal point of the cable can be found by inspection.

Problems

(*Solve by a direct attack. Do not use the equations above as formulas.*)

1. A steel cable weighing 1 lb per ft carries a lead-sheathed telephone cable weighing 11 lb per ft suspended so that the telephone cable is horizontal. Assume that the load is 12 lb per horizontal foot and find the tension at the middle and at the ends for a span of 400 ft with a sag of 20 ft.

Ans. $H = 12{,}000$ lb; $T = 12{,}238$ lb.

2. A steel cable is used to carry a horizontal pipe across a span of 500 ft. The pipe is horizontal and weighs (with cable and hangers) 40 lb per ft. The cable at the right end is 10 ft above the point to which the left end is attached and the lowest point of the cable is 8 ft below the left end. Find the maximum tension. Let H be the tension at the lowest point. Set up equations from both sides as free bodies in terms of H and the distances from the low point to the ends.

Ans. Maximum $T = 100{,}717$ lb.

3. A suspension bridge (without stiffening truss) has a span 300 ft long with the low point of the cable 100 ft below the tops of the towers. If the floor and hangers plus the weight of the cable amount to 3000 lb per ft and the center

100 ft of the span carries an additional load of 800 lb per ft, what is the tension at the middle of the cables and the slope at the upper ends? The bridge is supported by two cables symmetrically spaced. (If there were a stiffening truss, some of the central load would be spread out over the rest of the span in a manner not determinable by statics alone.)

4. When a simple uniform cable is suspended from two points at the same level and stretched rather tightly (*i.e.*, the sag is not more than one-tenth of the span) the center of gravity of half the cable is not far from the quarter point of the span, and a good approximate answer comes from making this assumption in problems like the following: A rope is stretched between two points at the same level and the sag is 24 ft. If the tension at the middle is equal to the weight of 300 ft of rope, what is the distance between the points? *Ans.* 240 ft, approximately.
5. A rope 240 ft long weighing 2 lb per ft is stretched between two points at the same height so that the tension in the middle of the rope is 800 lb. Assume that the span is 240 ft (approximately) and that the load is 2 lb per ft horizontally. Find the sag. *Ans.* 18 ft.
6. In Prob. 5 find the height above the lowest point of the rope at intervals of 24 ft horizontally from the middle of the rope, then assume the rope straight from point to point and find the approximate length of rope to reach across the 240-ft span. Compare the answers to this problem and the one above with those of Prob. 5 of Art. 107, which is the same problem by the exact method.

105. The Catenary. The curve followed by a uniform flexible chain or cord suspended from two points is called the *catenary*. (The word is derived from the Latin *catena*, a chain.) The problem of finding the equation of this curve and of finding the forces in the chain or cord is an important application of statics. For many practical purposes, the approximate methods indicated in the preceding article are sufficiently accurate. More exact solutions require the application of calculus or differential equations, but for certain types of problems the methods are relatively simple. The problems are all classified as nonconcurrent, coplanar and such of them as have three unknowns can be solved. If no force or unit weight is given, the forces at the various points on the cord can be found only in terms of the weight per unit length of the cord. The problems in this article and in the one following are those in which the unknowns are convenient to calculate. The more difficult forms, such as the case where the span and the length of the cable are given, are treated in the Appendix.

Figure 226 shows a flexible cord suspended from two points A and B. In this figure the points are at the same level. This, however, is not necessary. The cord weighs w pounds per unit of length. The force in the cord at every point is a tension parallel to its length. At the lowest point this tension is horizontal. The horizontal component of the tension is the same at all points. In the formulas which follow, this horizontal component is represented by *H*. It is customary to represent H/w by a single letter c.

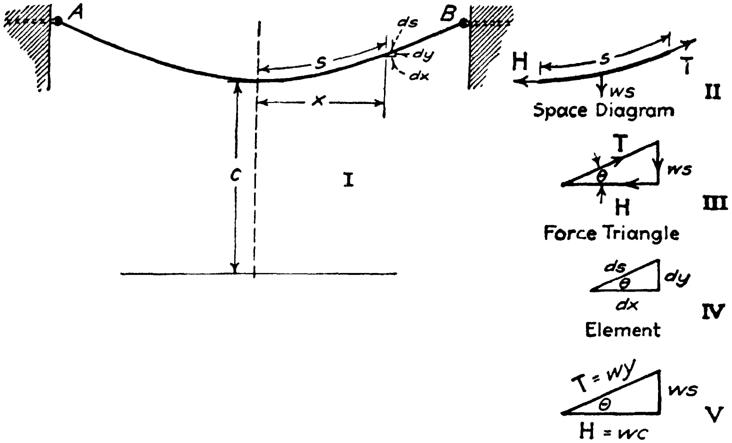


FIG. 226.

The constant c is the length of cord whose weight is equal to the horizontal tension (Fig. 227).

$$\frac{H}{w} = c \qquad H = wc$$

In Fig. 226, a portion of cord of length s is taken as the free body. The left end of the portion is at the lowest point of the curve, where the tension is horizontal. Figure 226,II shows this portion of the cord detached from the remainder. Three forces act on the portion. The horizontal tension **H** is toward the left at the lower end. The tension **T** in the direction of the tangent acts at the upper right end. The weight of the portion of cord is ws and acts vertically downward. Figure 226,III gives the force triangle for these forces. In Figure 226,I, an element of length ds is shown at the right end of the length s. The components of the element are dx and dy. Figure 226,IV shows the same element enlarged.

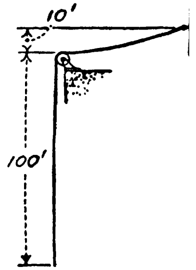


FIG. 227.

Since the tension at the end of the cord is along the tangent at that position, T is parallel to ds and the force triangle of Fig. 226,III is similar to Fig. 226,IV. These triangles furnish relations necessary for finding the equations of the catenary.

106. Deflections in Terms of Length. From the triangles of Fig. 226,

$$\frac{dy}{ds} = \frac{ws}{T} \tag{1}$$

This equation involves three variables, T, s, and y. The tension may be eliminated by substituting

$$T = \sqrt{w^2s^2 + H^2} \tag{2}$$

Equation (1) becomes

$$\frac{dy}{ds} = \frac{ws}{\sqrt{w^2s^2 + w^2c^2}} = \frac{s}{\sqrt{s^2 + c^2}} \tag{3}$$

Separating the variables in Eq. (3),

$$dy = \frac{s\,ds}{\sqrt{s^2 + c^2}} \tag{4}$$

Integrating,

$$y = \sqrt{s^2 + c^2} + K_1 \tag{5}$$

in which K_1 is an arbitrary constant of integration. If the origin is taken at the lowest point of the curve, so that $y = 0$ when $s = 0$, the value of K_1 is found to be equal to $-c$. It is customary to take the origin at a distance c below the lowest point of the curve, so that $y = c$ when $s = 0$. When these values are substituted in Eq. (5), then $K_1 = 0$ and Eq. (5) reduces to

$$y = \sqrt{s^2 + c^2} \tag{6}$$

$$y^2 = s^2 + c^2 \qquad \text{Formula IX}$$

A point on the curve at a distance s from the lowest point, measured along the curve, is at a vertical distance $y - c$ above the lowest point of the curve.

A right triangle may be drawn for which y is the hypotenuse and s and c are the sides. In Fig. 226,III, the altitude is ws and the base is H, which is equal to wc.

$$T^2 = w^2s^2 + w^2c^2 = w^2(s^2 + c^2) = w^2y^2 \tag{7}$$

$$T = wy \tag{8}$$

These relations are shown in Fig. 226,V. From this figure, it may be shown that

$$y = c \sec \theta \tag{9}$$

$$s = y \sin \theta \tag{10}$$

Example

A rope 100 ft long is stretched between points at the same level. The rope weighs 0.2 lb per ft and the horizontal component of the tension is 50 lb. Find the sag at the middle.

Since the supports are at the same level, the lowest point is at the middle. To find the elevation of the ends above the middle, the value of s is taken as one-half the length.

$$\begin{aligned} s &= 50 \text{ ft} \\ c &= \frac{50}{0.2} = 250 \text{ ft} \\ y^2 &= 50^2 + 250^2 \\ y &= 50\sqrt{26} = 254.95 \text{ ft} \\ \text{Sag} &= y - c = 254.95 - 250 = 4.95 \text{ ft} \end{aligned}$$

Problems

1. A chain 120 ft long, weighing 240 lb, is stretched between points at the same level. The horizontal tension is 896 lb. Find y, the sag, and the resultant tension at the ends. *Ans.* 452 ft; 4 ft; 904 lb.
2. A chain 120 ft long, weighing 240 lb, is stretched between points at the same level. The resultant tension at either end is 312 lb. Find the sag and the horizontal tension at the middle. *Ans.* 12 ft; 288 lb.
3. In Prob. 1, how far below the level of the supports is a point on the chain which is 20 ft from one end? *Ans.* 2.22 ft.
4. What is the slope at the ends of the chain of Prob. 1? *Ans.* 7°38′.
5. What is the slope of the chain of Prob. 2 at the ends and at 20 ft from either end? *Ans.* 22°37′; 15°31′.
6. A rope 100 ft in length is stretched between points at the same level and sags 10 ft at the middle. Find the length of rope whose weight is equal to the horizontal tension, and find the slope of the rope at the ends.

From Formula IX,

$$\begin{aligned} y^2 - c^2 &= 2500 \\ y - c &= 10 \\ y + c &= 250 \\ 2c = 240 \qquad c &= 120 \text{ ft} \\ \text{Slope} = {}^{50}\!/_{120} &= 0.4167 \end{aligned}$$

7. A rope is hung over two smooth pins, one of which is higher than the other by 5 ft. The rope hangs vertically downward from the lower pin 50 ft and is horizontal at the other point of contact with this pin. How far does it hang downward from the other pin and what is the total length of the rope? *Ans.* Length = 127.9 ft.
8. If the pins of Prob. 7 are perfectly smooth, is the equilibrium stable? What is the force exerted by the lower pin? *Ans.* Not stable. Force at lower pin equals weight of 70.7 ft of rope.
9. A chain 100 ft long, weighing 3 lb per ft, is stretched between points at the same level. The horizontal component of the tension is 600 lb. Find the direction of the chain and the resultant tension at the ends and at 20 ft from either end, measured along the chain. Solve by the force triangle (Fig. 226,III) without using any of the results of Formula IX.

10. A chain 40 ft long weighing 4 lb per ft is attached to a point A and extends to a point B, which is 8 ft lower than A. At B a rope weighing 1 lb per ft is attached to the chain. It is desired to attach the rope to a point C on the same elevation as A and to have the rope and chain horizontal at B. What length of rope is required? *Ans.* 78.8 ft.

11. Point B is 4 ft higher than point A. A chain hung from A to B has its lowest point 2 ft below A. Find c if the length of the chain is 28 ft. *Ans.* 24 ft.

107. Deflection in Terms of Span. The equations of Art. 106 were derived from the relation of dy to ds of Fig. 226,IV. To get the deflection in terms of horizontal distance, the relation of dy to dx is used. From the similar triangles III and IV of Fig. 226,

$$\frac{dy}{dx} = \frac{ws}{H} = \frac{s}{c} \tag{1}$$

Substituting $e = \sqrt{y^2 - c^2}$ to reduce the variables to two,

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c} \tag{2}$$

$$\frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c} \tag{3}$$

Integrating Eq. (3)

$$\log\,(y + \sqrt{y^2 - c^2}) = \frac{x}{c} + K_2 \tag{4}$$

It has already been assumed that $y = c$ at the lowest point. If the Y axis is taken through the lowest point, then $x = 0$ when $y = c$. Substituting in Eq. (4),

$$\log c = K_2$$

and the equation becomes

$$\log \frac{y + \sqrt{y^2 - c^2}}{c} = \frac{x}{c} \tag{5}$$

$$x = c \log_e \frac{y + s}{c} \qquad \text{Formula X}$$

When Eq. (5) is changed from a logarithmic to an exponential function, it becomes

$$\frac{y + \sqrt{y^2 - c^2}}{c} = e^{x/c} \tag{6}$$

$$y + \sqrt{y^2 - c^2} = ce^{x/c} \tag{7}$$

$$y^2 - c^2 = c^2e^{2x/c} - 2cye^{x/c} + y^2 \tag{8}$$

$$y = \frac{c}{2}\,(e^{x/c} + e^{-x/c}) \qquad \text{Formula XI}$$

Formula XI is the principal equation of the catenary. Differentiating Formula XI,

$$\frac{dy}{dx} = \frac{e^{x/c} - e^{-x/c}}{2} \tag{9}$$

From Eq. (1),

$$s = c\frac{dy}{dx} = \frac{c}{2}\left(e^{x/c} - e^{-x/c}\right) \qquad \text{Formula XII}$$

The student who is familiar with hyperbolic functions will recognize $(e^{x/c} + e^{-x/c})/2$ as the hyperbolic cosine of x/c and $(e^{x/c} - e^{-x/c})/2$ as the hyperbolic sine of x/c.

Formulas XI and XII may be written,

$$y = c \cosh \frac{x}{c} \qquad \text{Formula XIII}$$

$$s = c \sinh \frac{x}{c} \qquad \text{Formula XIV}$$

Example 1

A rope weighing 0.1 lb per ft is stretched between points which are 120 ft apart and at the same level. The horizontal tension is 50 lb. Find the length of the rope and the sag.

This is a problem in which x and c are given: $x = 60$ ft; $c = 500$ ft.[1]

$$y = 500 \times \cosh 0.12 = 500 \times 1.0072 = 503.6 \text{ ft} \qquad \text{sag} = 3.6 \text{ ft}$$
$$s = 500 \times \sinh 0.12 = 500 \times 0.1203 = 60.15 \text{ ft}$$
$$\text{Length} = 2s = 120.3 \text{ ft}$$

Example 2

Find the horizontal distance between the ends of the chain of Prob. 1 of Art. 106.

$$x = 448 \log_e \frac{452 + 60}{448} = 448 \log_e \frac{5.12}{4.48}$$
$$\log_e 5.12 = 1.63315$$
$$\log_e 4.48 = \underline{1.49962}$$
$$0.13353$$
$$x = 448 \times 0.13353 = 59.821$$
$$2x = 119.64 \text{ ft}$$

Problems

1. Find the horizontal distance between the ends of the chain of Prob. 2 of Art. 106. *Ans.* $2x = 228 \times 0.40547 = 116.78$ ft.

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[1] Table 2 on page 405 gives the hyperbolic functions from $x = 0.080$ to $x = 0.120$. Complete tables are found in some books of trigonometric and logarithmic functions.

2. A cable 88 ft long is stretched between points at the same level and sags 4 ft. Find the horizontal distance between the ends. *Ans.* 87.51 ft.
3. A chain weighing 2 lb per ft is stretched between points at the same level 180 ft apart. The horizontal tension is 1600 lb. Find the length of the chain and the sag. *Ans.* 180.37 ft; 5.07 ft.
4. A cable weighing 1.6 lb per ft is stretched between points at the same level 600 ft apart. The horizontal tension is 2400 lb. Find the length, the sag, and the resultant tension at the ends.
5. A rope 240 ft long weighing 2 lb per ft is stretched between points at the same level. If the horizontal tension is 800 lb, find the sag and the span.
   Ans. Sag = 17.6 ft; span = 236.52 ft, or 3.48 ft less than the length of the rope. (Compare Probs. 5 and 6 of Art. 104.)
6. Find the horizontal distance between the ends of the chain of Prob. 11, Art. 106.
7. A chain 150 ft long weighs 180 lb. The resultant tension at one end of the chain is 240 lb and at the other is 264 lb. What is the difference of height of the two ends? How far is the lowest point below the lowest end? What is the horizontal distance between the two ends? *Ans.* 20 ft; 5.60 ft; 145.25 ft.
8. The chain of Prob. 7 is supported at two points at the same level. When a load of 240 lb is hung on the middle of the chain the resultant tension at each end is 750 lb. Find the distance of the center below the supports. Find the horizontal distance between the supports. *Ans.* 16.724 ft; 146.13 ft.
9. A chain 200 ft long, weighing 200 lb, is stretched between points at the same level. A load of 110 lb is hung 60 ft from each end of the chain and the resultant tension at either end is found to be 290 lb. Find the horizontal distance between the loads. (Problems of this type can best be solved by replacing the concentrated loads by equivalent lengths of chain of the same unit weight as the actual chain.)
10. Solve Prob. 9 with one of the 110 lb loads removed, assuming the 290 lb as the tension at the end from which the load has been removed. (Removing the load changes the tension at both ends.)

CHAPTER 12

CENTER OF GRAVITY

108. Center of Gravity and Mass Center. The *center of gravity* of a body is a point through which a single force must act to support the body in the field of gravity regardless of the orientation of the body. The *mass center* of a body is the point through which the resultant of all forces must act when a system of forces produces a linear acceleration of the body without producing any angular acceleration. The terms are usually regarded as synonymous in mechanics. The mass center of a right circular cone is on the axis at a distance from the base equal to one-fourth of the height. If such a cone is suspended by a cord so that a bottom element is horizontal, the point where the line of the cord cuts the axis of the cone would be a little toward the vertex from the center of mass since the gravitational attraction of an element of mass at the top of the cone near the base is a little less than that for an equal element of mass on top near the vertex, because of the difference in distance from the center of the earth, but the difference would be almost inconceivably minute. Also, if the top element of the cone is made horizontal, it could be argued that the center of gravity shifts the other way. It is an academic question. The term center of gravity is usually used when *mass center* is meant. Precisely stated, the center of gravity is a variable point never very far from the mass center. It has been assumed that the student is familiar with the meaning of center of gravity, and the term has been freely used in the preceding chapters.

109. Center of Gravity by Experiment. A single flexible cord from which a body is hung points toward the center of gravity of the body. The extremely complicated system of forces exerted on the body by gravitational attraction is imagined to be replaced by the resultant of this system, and this resultant is in equilibrium with the force of the cord. The two are equal, opposite, and along the same line. By suspending the body successively from three different points, three lines are obtained which intersect at the center of gravity.

When the body is so large that it cannot conveniently be suspended, it may be supported in some statically determinate manner, the reactions weighed on platform or other scales, and the center of gravity determined by a reverse operation of determining the reactions required to support a body of known weight and known center of gravity.

The coordinates of the center of gravity are represented by $\bar{x}$, $\bar{y}$, and $\bar{z}$. These may be read "bar x," "bar y," and "bar z."

Example

The base ABC of Fig. 228 is a plane surface and the faces ABDE and DBC are also plane and are perpendicular to each other and to the base. The body weighs 24 lb. The body rests on three supports, which are not shown in the drawing. Two supports are 1 in. from the edge AB. The third support, 17 in. from the edge AB, rests on a platform scale. The scale reading is 4.52 lb when the base is horizontal. How far is the center of gravity from the plane ABDE?

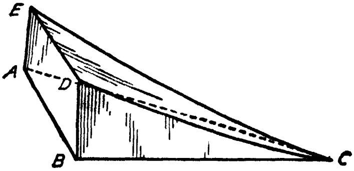

FIG. 228.

Taking moments about the two supports,

$$24\bar{x} = 4.52 \times 16$$
$$\bar{x} = 3.01 \text{ in.}$$

The center of gravity lies in a vertical plane which is parallel to the face *ABDE* and is 4.01 in. from it.

When the body is supported on two points, each 1 in. from the edge *BC*, and a third point rests on a platform scale 13 in. from *BC*, the scale reading is found to be 7.24 lb. From this experiment, the center of gravity is found to lie in a plane parallel to the face *BCD* at a distance of 4 62 in. from that plane. The intersection of these two planes through the center of gravity gives the location of a line through the center of gravity.

To find the location of the center of gravity on the line of intersection of the two planes, its position must be changed so that the forces of gravity will act in a relatively different direction. For the experiment the body is placed with the face *BCD* horizontal. When it is supported at two points, each 1 in. from *BC*, and on a third point at a distance of 11 in. from *BC*, the scale supporting the third point reads 7.20 lb. From this reading the center of gravity is found to be 4.00 in. from the plane *ABC*.

It will be observed that each of these experiments locates a vertical plane which is parallel to the axis of moments and passes through the center of gravity. Three such planes, if no two of them are parallel, fully locate the center of gravity.

Problems

1. A triangular board of uniform thickness is 12 in. wide at the base. The perpendicular distance from the vertex to the base is 18 in. The board is supported on two knife-edges, each 0.5 in. from the base, and on a third knife-edge, parallel to the base, at 1 in. from the vertex. The board weighs 15.00 lb. When the

third knife-edge, which weighs 0.45 lb, rests on a platform scale, the reading is 5.45 lb. How far is the center of gravity of the board from the base?

Ans. 6.00 in

2. A rectangular plate of uniform thickness is 24 in. long and 18 in. wide. A circular hole, 10 in. in diameter, is cut from the plate. The center of the hole is 6 in. from the front 24-in. edge and 8 in. from the left 18-in. edge. The plate is supported on a knife-edge 0.1 in from the left edge and on a second knife-edge at the same distance from the right edge. The knife-edge at the right rests on a platform scale. The increase of the load on the scale when the plate is put on the knife-edge amounts to 22.72 lb. The plate weighs 42.40 lb. How far is its center of gravity from the left edge?

As a second experiment, the plate is supported on a knife-edge 0.1 in. from the front and on a second knife-edge 17 in. from this one. The load on the second knife-edge reads 23.87 lb. How far is the center of gravity from the front edge of the plate?

Figure 229 shows a convenient method of determining the center of gravity of small bodies. The body whose center of gravity is desired is placed on a balance beam and carefully balanced with the beam in a horizontal position. It is then turned 180° about a vertical axis to the position shown by the broken lines of the figure, and moved along the beam until exact balance is obtained with no change of the weights. The center of gravity is then at the same location as it was in the first position. If C is a point in the body and C' is the same point after the body is turned, the center of gravity is located in a vertical line midway between C and C'.

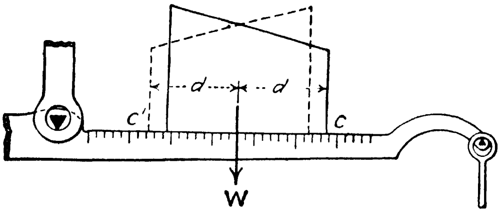

FIG. 229.

110. Center of Gravity Geometrically. *When the density of a body is uniform,* the center of gravity may sometimes be determined geometrically.

Since every element of a body at right angles to a plane of symmetry extends an equal distance on each side of the plane, the moment of the element about any axis in the plane is zero. Since the body is made up of a series of elements, each of which has zero moment with respect to any axis in the plane of symmetry, the moment of the entire body is zero. *If a body of uniform density has a plane of symmetry, the center of* *gravity lies in that plane.* If there are two or more planes of symmetry, the center of gravity lies at their intersection.

All planes through the center of a sphere are planes of symmetry. *The center of gravity of a sphere of uniform density is at its center.*

All planes through the axis of a right circular cylinder are planes of symmetry. The plane perpendicular to the axis at the middle is a plane of symmetry. *The center of gravity of a right circular cylinder of uniform density lies at the middle of the axis.*

The major and minor axes of an ellipse are axes of symmetry. *The center of gravity of a right elliptical cylinder of uniform density lies at the middle of the axis.*

The center of gravity of a rectangular parallelepiped of uniform density is at a point which is midway between any pair of opposite faces.

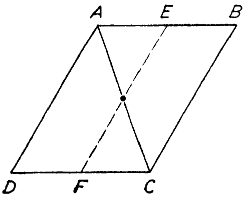

FIG. 230.

The center of gravity of a right prism of uniform density, the sections of which are equilateral triangles, is located at the middle of its length at the intersection of the lines which bisect the angles of the section.

Figure 230 shows an oblique-angled parallelogram which represents one end of a parallelepiped with rectangular sections perpendicular to the plane of the paper. There is no axis of symmetry. If the half *EBFC* were rotated 180° around the line *EF*, it would not coincide at the ends with *AEFD*. Neither would the triangle *ABC* coincide with *ACD* if rotated about *AC*. If, however, the triangle *ABC* is rotated 180° about an axis *perpendicular* to the plane of the paper through the middle of *AC*, the line *AB* coincides with *CD* and the line *CB* coincides with *AD*. The moment of *ABC* with respect to the axis *AC* is equal and opposite to the moment of *ACD*. The center of gravity of any parallelogram lies at the intersection of the diagonals, or midway between the parallel faces. The center of gravity of any oblique prism or cylinder of uniform density lies in a plane parallel to the bases and midway between them.

111. Center of Gravity of a Triangular Plate.

Figure 231 represents a triangular plate of uniform thickness perpendicular to the plane of the paper. *OH* is the median line. The triangle is divided into elements such as *ADFB*, by lines parallel to the base. The moment of the oblique parallelogram *ACED* with respect to the line

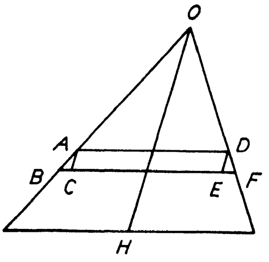

FIG. 231.

OH (or any line in the plane through *OH* perpendicular to the plane of the paper) is zero. The small triangles *ABC* and *DEF* are left over. These triangles have equal areas, and their bases are equally distant from the line *OH*; *consequently, their moments about OH* must be *approximately* equal. The lines *AD* and *BF* may be brought infinitely close together, so that the areas of these triangles become infinitesimals of the second order, and the differences of their moments become infinitesimals of the third order. The center of gravity of the element *ADFB* must then lie in the median line. The entire triangle is made up of similar elements, each of which has its center of gravity on the median line. It follows that the center of gravity of the entire triangular plate of uniform thickness and density must lie on the plane of the median line.

A second median may be drawn as in Fig. 232. The center of gravity lies in this median, and, consequently, at the intersection of the medians.

It was learned in geometry that the intersection of the medians of a triangle is located at one-third the length from the base. In Fig. 232, *HK* is one-third of *OH*, and *KL* is one-third of *ML*. If h is the perpendicular distance from a vertex to the opposite base, and $\bar{y}$ is the perpendicular distance from the center of gravity to that base, then $\bar{y}$ is one-third of h.

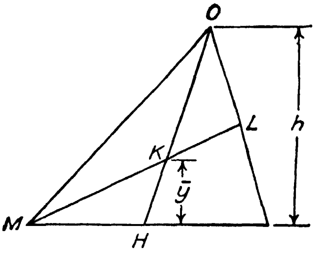

FIG. 232.

112. Calculation of Center of Gravity by Moments. When the location of the center of gravity is known for each of the parts which make up a body, the center of gravity of the body as a whole is calculated by moments. The problem is that of finding the position of the resultant of parallel forces which are proportional to the masses of the separate parts of the body.

To find $\bar{x}$ experimentally, the body was placed with the *XZ* plane horizontal, and moments were measured with respect to the *Z* axis, or some axis parallel to it. To find $\bar{z}$, moments were measured with respect to the *X* axis. To find $\bar{y}$, the body was turned through a right angle about the *Z* axis or the *X* axis and moments were measured with respect to that axis. To calculate $\bar{y}$, it is not necessary to consider the position of the body as changed. The forces proportional to the masses of the separate parts may be regarded as acting parallel to the *XZ* plane and moments computed with respect to some axis in that plane.

The moment of a force is the same for any axis perpendicular to its direction in a plane which is parallel to it. For instance, the moment of a vertical force is the same for all horizontal axes in a given vertical plane. It is customary, therefore, in finding the center of gravity or other resultants of parallel forces to speak of *moment with respect to a plane* instead of moment with respect to an axis in the plane.

The distance of the center of gravity of a system of bodies from a reference plane is the sum of the moments with respect to the plane divided by the sum of the masses.

$$\bar{x} = \frac{m_1x_1' + m_2x_2' + m_3x_3' + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma mx'}{\Sigma m} \qquad \text{Formula XV}$$

in which $m_1, m_2, \ldots,$ are the masses and $x_1', x_2', \ldots,$ are the respective distances from the reference plane. If the density is the same for all the bodies, the masses are proportional to the volumes and Formula XV may be written

$$\bar{x} = \frac{\Sigma Vx'}{\Sigma V} \tag{1}$$

If the masses are plates of the same thickness and density, the area may be used instead of the volume and Formula XV becomes

$$\bar{x} = \frac{\Sigma Ax'}{\Sigma A} \tag{2}$$

The centers of gravity of other geometric figures can be found by geometry but the proofs are often laborious. The locations of the centers of gravity of the three important geometrical figures given below are to be derived in Art. 118 by the method of integral calculus.

The center of gravity of a pyramid lies on the line from the vertex to the center of gravity of the lamina constituting the base at one-fourth of the length of that line from the base.

The center of gravity of a cone is at one-fourth of the height from the base.

The center of gravity of a solid hemisphere is at three-eighths of the radius from the diametral plane which bounds it.

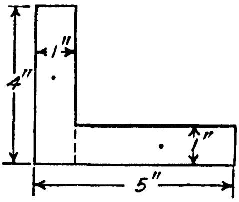

FIG. 233.

Problems

1. A plate of uniform thickness and density is of the form of a 5- by 4- by 1-in. angle section as shown in Fig. 233. Find the distance of the center of gravity from the back of each leg. Solve by dividing the section into two equal rectangles. Check $\bar{x}$ by dividing the section into a 5- by 1-in. rectangle and a 1- by 3-in. rectangle. Since the thickness and density are constant, the terms which represent them may be omitted.

Taking moments about the back of the 4-in. leg,

Area	Arm	Moment
4	0.5	2.0
4	3.0	12.0
8		14.0

Total area = 8 total moment = 14

$$8\bar{x} = 14$$

$$\bar{x} = 1.75 \text{ in.}$$

2. Find the distance of the center of gravity of the plate of Prob. 1 from the back of the 5-in. leg.

3. Solve Probs. 1 and 2 by subtracting the area and moment of a 4- by 3-in. rectangle from the area and moment of the 5- by 4-inch rectangle.

4. A board of uniform thickness and density consists of a rectangle 8 in. wide and 5 in. high, on top of which stands a triangle 8 in. wide at the base and 6 in. high. Find the distance of the center of gravity from the base of the rectangle.
Ans. $\bar{y} = 4.1875$ in.

5. The base of the triangle of Prob. 4 coinicides with the top of the rectangle. The vertex is 1 in. to the left of the vertical line along the right edge of the rectangle. Find the distance of the center of gravity from the center line of the rectangle. Solve by means of the geometrical position of the center of gravity of the triangle. Solve again by subtracting the moments of the two right triangles which make the area a rectangle. *Ans.* $\bar{x} = 0.375$ in.

6. A rectangular parallelepiped is 8 in. square and 6 in. high. On top stands a pyramid 12 in. high. The base of the pyramid coincides with the top of the parallelepiped. The vertex is 2 in. to the right of the vertical line through the center of the base. How far is the center of gravity from the base of the parallelepiped, and how far from the vertical plane along the left face?
Ans. $\bar{y} = 5.4$ in.; $\bar{x} = 4.2$ in.

7. A plate of uniform thickness and density is in the form of a trapezoid. The lower base is 18 in., the upper base is 10 in., and the height is 12 in. Find the distance of the center of gravity from the lower base. Solve by dividing the trapezoid into two triangles, as in Fig. 234,II, and also by dividing into a parallelogram and a triangle, as in Fig. 234,I.

8. Solve Prob. 7 by extending the sides of the trapezoid upward until they meet to form a triangle.

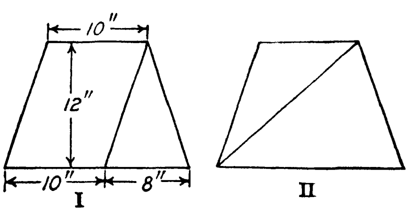

FIG. 234.

9. A board of uniform density and thickness consists of a rectangle 12 in. wide and 10 in. high with a circular hole 6 in. in diameter. The axis of the hole is 4 in. from the base and 4 in. from the left edge. Find the center of gravity.
Ans. $\bar{x} = 6.616$ in.; $\bar{y} = 5.308$ in.

10. A circular plate of uniform thickness and density is 20 in. in diameter. A circular hole 6 in. in diameter has its center 4 in. to the right of the center of the plate. Find the center of gravity.

Ans. $\bar{x} = {}^{36}/_{91} = 0.3956$ in. to the left of the center.

11. A uniform solid cylinder, 10 in. in diameter and 8 in. high, has a cylindrical cavity 4 in. in diameter and 4 in. deep in one end. How far is the center of gravity of the remainder from the other end? *Ans.* 3.826 in.

12. A body of uniform density consists of a cylinder 8 in. in diameter and 10 in. high, and a hemisphere 12 in. in diameter. The center of the hemisphere coincides with the upper end of the axis of the cylinder. Find the center of gravity. (See example of Art. 118.) *Ans.* $\bar{y} = 8.434$ in. from the base.

13. A cylinder of diameter d and height h has a conical cavity of diameter d at the base and height h. Find the distance of the center of gravity of the remainder from the top. *Ans.* $\bar{y} = 5h/8$.

14. A frustum of a cone has the lower base 8 in. in diameter, upper base 6 in. in diameter, and height 12 in. Find the center of gravity.

15. A hemisphere 10 in. in diameter has a hemispherical cavity 6 in. in diameter. Find the distance of the center of gravity from the plane which bounds the solid. *Ans.* 2.081 in.

113. Center of Gravity of an Area. The center of gravity of an area is calculated as though it were a thin plate with one unit of mass per unit of area. Purists in speech, noting that gravity cannot act upon a geometric figure, insist upon calling the center of gravity of such figures (both areas and solids) *centroids*, and this word is in regular and general use. The centroid of the cross-section area of a structural member subjected to bending moment is an important factor in the problem of calculating the unit stresses in the member. The total pressure on any plane area resulting from a distributed pressure with an intensity varying directly with the distance from some line, such as the pressure of a liquid, is found by multiplying the area by the unit pressure at the centroid of the area. These, and numerous other applications, make the problem of finding centroids of plane areas of great practical importance, especially the problem of finding the centroid of combinations of two or more standard rolled sections.

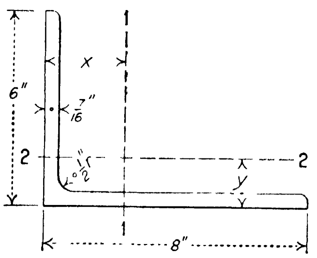

FIG. 235.

Figure 235 represents the cross section of an 8- by 6- by $\frac{7}{16}$-inch angle section.[1] The axis 1-1 and the axis 2-2 intersect at the center of gravity and are parallel to the backs of the legs. The figure shows the fillet at the junction of the legs and the radii at the inner corners of the outer ends of the legs. In calculating the center of gravity and the moment of inertia of the section, these curves are disregarded, and the sections are treated as though they were made up of two rectangles, as shown in Fig. 233. For use in the following problems and some later ones, the student should have a steel construction handbook or some other book giving dimensions and properties of the standard rolled-steel sections. There are also standard aluminum sections.

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[1] In the Steel Construction Manual of the American Institute of Steel Construction, the figures for angles of unequal legs are shown with the longer leg vertical, so that x and y are interchanged.

Problems

1. Find x and y for Fig. 235. Check and compare with the handbook.

2. Find the distance of the center of gravity of an 8- by 6- by 1-in. angle section from the back of the longer leg. Check by subtracting the area and moment of a 7- by 5-in. rectangle from the area and moment of an 8- by 6-in. rectangle. Compare with the handbook.

3. Find the distance of the center of gravity of the angle of Prob. 2 from the back of the shorter leg.

4. Find the area and the location of the center of gravity of a 5- by 3½- by ½-in. angle section. Check by using a section of twice these dimensions.

5. Find the location of the center of gravity of a 4- by 3½- by $\frac{5}{16}$-in. angle section.

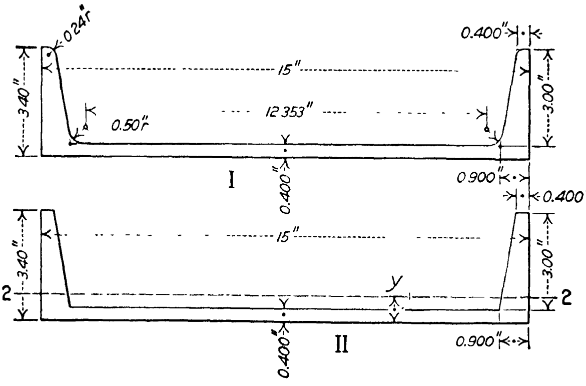

FIG. 236.

Figure 236 shows a 15-inch 33.9-pound standard channel section. The upper figure shows the section as rolled. The lower figure shows the section as calculated with the data required for computing the center of gravity and moment of inertia. Axis 2-2 passes through the center of gravity of the section at a distance y from the back of the web. The calculation includes the moment of three areas. These are a rectangle which includes all the web, a rectangle in each flange which may be combined into one, and a triangle in each flange which may be combined into one.

Heavier channels of the same number are made by increasing the thickness of the web, with no change in the flange except for the width.

Problems

6. Find the distance of the center of gravity from the back of the web for the 15-in. 33.9-lb channel section of Fig. 236. *Ans.* 0.794 in.

7. Find the distance of the center of gravity from the back of the web for a 15-in. 40-lb channel section. Compare with the handbook. Use as much of the work of Prob. 6 as possible. *Ans.* 0.783 in.

8. Look up the dimensions and calculate the distance of the center of gravity from the back of the web for an 8-in. 11.5-lb. channel section.

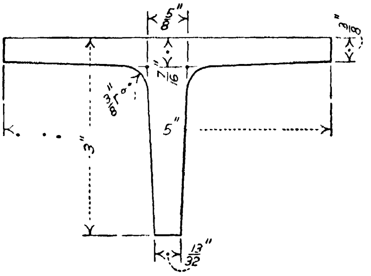

Fig. 237.

Figure 237 shows a Carnegie T section, T50. The flange is 5 inches and the stem is 3 inches. The flange thickness is $\frac{3}{8}$ to $\frac{7}{16}$ inch. The stem thickness is $\frac{13}{32}$ to $\frac{5}{8}$ inch. The section is made up of three rectangles and four triangles. The two triangles in the flange may be taken together as one triangle $4\frac{3}{8}$ inches long and $\frac{1}{16}$ inch high. The two triangles in the stem are equivalent to a single triangle $3\frac{9}{16}$ inches high and $\frac{7}{32}$ inch wide.

Problem

9. Find the area and the distance of the center of gravity from the top of the flange for the T section of Fig. 237. *Ans.* $\frac{3453}{1024} = 3.37$ sq in.; 0.763 in.

114. Fabricated Sections. An engineer frequently calculates the position of the center of gravity of sections which are built up of two or more rolled shapes riveted or welded together. The area and the location of the center of gravity of each member are taken from the handbook, and the problem is that of finding the constants of the combination.

Problems

1. A 12- by ½-in. plate is riveted to the lower flange of an 8-in. 19.7-lb wide-flange section. Find the distance of the center of gravity of the section from the bottom of the plate. Check.

2. An 18-in. 54.7 lb *I* beam has two 6- by 4- by 3/8-in. angles riveted to the web, as in Fig. 238. The 6-in. legs of the angles are perpendicular to the web of the *I* beam, and the tops of the legs are 12 in. below the top of the upper flange of the beam. Find the center of gravity.
3. Two 12-in. 35-lb channels are placed 9 in. back to back and a 16- by ½-in. plate is riveted or welded to the two lower flanges. Find the distance of the center of gravity of the section from the bottom of the plate.

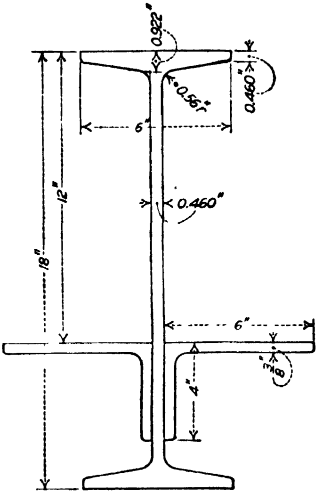

FIG. 238.

115. Center of Gravity by Integration.

When forces are parallel, the moment of any force with respect to an axis perpendicular to the direction of the forces is equal to the product of the magnitude of the force and its distance from a plane which includes the axis and is parallel to the force. In Art. 112 this product was called *moment with respect to a plane*. In that article was given a formula for the center of gravity of a composite body,

$$m_1x_1 + m_2x'_2 + m_3x'_3 + \cdots = \bar{x}(m_1 + m_2 + m_3 + \cdots),$$

which can best be written (Formula XV):

$$\bar{x} = \frac{\Sigma mx'}{\Sigma m}$$

When the elements are infinitesimal, *m* is replaced by *dm* and the summation sign is replaced by the integration sign.

$$\text{Total mass} = \int dm$$

$$\text{Total moment} = \int x'\,dm$$

$$\bar{x} = \frac{\int x'\,dm}{\int dm} = \frac{\int x'\,dm}{\text{total mass}} \qquad \text{Formula XVI}$$

If ρ is the density, the mass of an element is $\rho\,dV$, and its moment is $x'\rho\,dV$, Formula XVI becomes

$$\bar{x} = \frac{\int \rho x'\,dV}{\int \rho\,dV}$$

If the density is constant throughout the body, ρ may be canceled before integrating and Formula XVI may be written

$$\bar{x} = \frac{\int x' \, dV}{\int dV} \qquad (1)$$

For a plate of uniform thickness and density,

$$\bar{x} = \frac{\int x' \, dA}{\int dA} \qquad (2)$$

in which dA is an element of area.

$$\bar{y} = \frac{\int y' \, dm}{\int dm} \qquad (3)$$

$$\bar{z} = \frac{\int z' \, dm}{\int dm} \qquad (4)$$

It must be remembered that x' is the distance of the center of gravity of the element from the plane of reference. When the plane of reference does not pass through the origin of coordinates, or when the element has finite dimensions in the direction of the X axis, the moment arm x' is usually different from the coordinate x.

116. Center of Gravity by Single Integration. The center of gravity of a uniform plate is found by Eq. (2) of the preceding article. The so-called *center of gravity of an area*, which is an important factor in the theory of beams, is calculated as if the area were a uniform thin plate.

The expression for the element often contains constants which appear in both the integral for the moment and the integral for the area. Since it is often desirable to check the expression for the element and the limits of integration by comparison with the total area computed, either accurately or approximately, in another way, these constants should not be canceled until after the limits have been substituted in the integrals. For the same reason, variables in the integrals should not be canceled until after the limits have been substituted in problems in which one limit is zero. When neither limit is zero, cancellation before substitution gives erroneous results.

Example

Find the distance of the vertex of a triangle from the line which is parallel to the base and passes through the center of gravity of the area (Fig. 239).

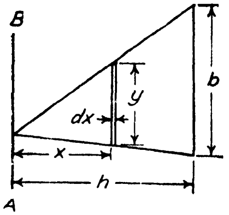

FIG. 239.

The element of area is dx wide and y high. This form is chosen because the entire triangle may be built up of such elements, and because the moment arm is easy to find.

$$dA = y\,dx \qquad x' = x^{*}$$
$$\text{Moment} = xy\,dx \tag{1}$$

The expression for the moment, Eq. (1), contains two variables x and y, one of which must be expressed in terms of the other before integration. From the similar triangles of Fig. 239,

$$\frac{x}{y} = \frac{h}{b} \tag{2}$$

$$y = \frac{bx}{h} \tag{3}$$

$$M = \int \frac{bx^2}{h}\,dx = \frac{b}{h}\int x^2\,dx = \frac{b}{h}\left(\frac{x^3}{3}\right) \tag{4}$$

The limits of $x = 0$ and $x = h$.

$$M = \frac{b}{h}\left[\frac{x^3}{3}\right]_0^h = \frac{bh^2}{3}$$

Dividing the moment by the area,

$$\bar{x} = \frac{bh^2}{3} \times \frac{2}{bh} = \frac{2h}{3}$$

Problems

1. Instead of expressing y in terms of x in the example, express x and dx in terms of y and dy and integrate between the proper limits. Also integrate $y\,dx$ for the area with x as the independent variable. Integrate $y\,dx$ with y as the independent variable.
2. Find $\bar{x}$ for the area bounded by the X axis, the curve $y^2 = 4x$, and the line $x = 9$. Use a vertical element as in Fig. 240 with x as the independent variable.

$$\textit{Ans. } M = 2\int_0^9 x^{3/2}\,dx = \frac{972}{5}$$

$$A = 2\int_0^9 x^{1/2}\,dx = 36$$

$$\bar{x} = \frac{\frac{972}{5}}{36} = 5.4$$

---

* The entire area is a parallelogram of area $y\,dx$ plus a small triangle. The area of this triangle is $(b\,dx^2)/2h$, which is a multiple of the second power of dx. The moment arm of the parallelogram is $x + (dx/2)$ and its moment is $xy\,dx + (y\,dx^2)/2$. The moment of the triangle is composed of terms of the second and third powers of dx. For integral calculus, all differentials above the first degree are neglected (see any calculus). For a proof that these higher powers may be neglected when the increment is not infinitesimal, see *Proc. Soc. Promotion Eng. Education*, vol. XXIV, 1916, pp. 319–327.

3. Solve Prob. 2 for $\bar{x}$ with y as the independent variable.
4. Find $\bar{y}$ for the area of Fig. 240, using a vertical element with x as the independent variable.
5. Find $\bar{y}$ for the area of Fig. 240, using a vertical element with y as the independent variable.
6. Find $\bar{x}$ for the area bounded by the X axis, the curve $y^2 = 4x$, and the lines $x = 4$ and $x = 9$ (Fig. 240).
7. Solve Prob. 6 for $\bar{y}$.
8. Find $\bar{y}$ for the area bounded by the X axis, the curve $y^2 = 4x$, and the line $x = 9$, using a horizontal element as in Fig. 241.

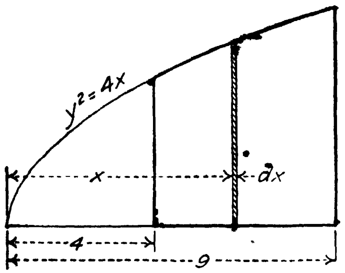

FIG. 240.

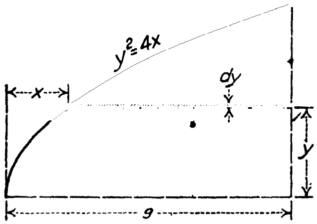

FIG. 241.

9. Solve Prob. 2 for $\bar{x}$, using the horizontal element of Fig. 241.
10. Find $\bar{y}$ for the area bounded by the curve $y^2 = 4x$, the line $x = 9$, and the line $y = 4$. Use a horizontal element. *Ans.* $\bar{y} = \frac{25}{16/3} = 4.6875$.
11. Solve Prob. 10 for $\bar{x}$. *Ans.* $\bar{x} = 7.275$.
12. Check Probs. 10 and 11 approximately by means of the triangle whose sides are the line $x = 9$, the line $y = 4$, and the straight line which joins the intersections of the first two lines with the parabola.

117. Center of Gravity by Double Integration. For double integration, the element is infinitesimal in two directions. The element of area in rectangular coordinates is $dA = dx\,dy$.

$$A = \int dA = \int\int dx\,dy \text{ or} \int\int dy\,dx \qquad (1)$$

depending upon which variable is integrated first.

$$\int\int dx\,dy = \int\left(\int dx\right) dy \qquad (2)$$

i.e., the second integral sign applies to the differential which immediately follows it, and the first integral sign applies to the entire expression after the term in the parentheses of Eq. (2) has been integrated between the proper limits. (Some textbooks of calculus apply the first integral sign to the first differential. Either method may be used if followed consistently.)

To find the area between the curve $y = x^2$ and the straight line $y = 2x$ of Fig. 242,

$$A = \int\int dx\,dy = \int (x)\,dy \tag{3}$$

The integration with respect to x consists of placing rectangles dx wide and dy high on the horizontal line from D to E of Fig. 242,I. These rectangles build up a horizontal element of length DE and height dy extending from the straight line to the curve. The limits of the first integration must give the length of *any* such horizontal element, hence the lower limit is the value of x at the straight line and the upper limit is the value of x at the parabola, both expressed in terms of y.

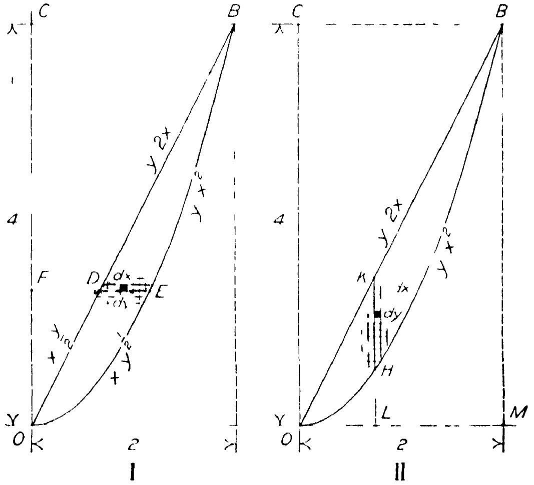

FIG. 242.

$$A = \int \Big[x\Big]_{y/2}^{y^{1/2}} dy = \int \left(y^{1/2} - \frac{y}{2}\right) dy = \left[\frac{2}{3}y^{3/2} - \frac{y^2}{4}\right] \tag{4}$$

The integration with respect to y (if dx has been integrated first) consists in piling up elements similar to DE from O to B to fill out the required area. The limits of y are 0 and 4.

$$A = \left[\frac{2}{3}y^{3/2} - \frac{y^2}{4}\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

The coefficient of dy in Eq. (4), which is $y^{1/2} - y/2$, is the length of the element. This could be calculated without integrating and the area could be found by a single integration. *A solution by a single integration is equivalent to the second step of a double integration.* When the first step can be made conveniently without integrating, double integration is unnecessary.

Figure 242,II shows the geometry of the integration when dy is integrated first. The rectangular elements of area dy times dx are piled into a vertical strip from H to K. The lower limit is the value of y at the parabola and the upper limit is the value of y at the straight line. For the second integration, these vertical strips are placed side by side from $x = 0$ to $x = 2$.

$$\int\int dy\,dx = \int \Big[y\Big]_{x^2}^{2x} dx = \int (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{4}{3} \qquad (5)$$

The equation for finding the moment with respect to the Y axis, when x is integrated first, is

$$M = \int x\,dA = \int\int x\,dx\,dy = \int \left[\frac{x^2}{2}\right]_{y/2}^{y^{1/2}} dy \qquad (6)$$

$$M = \int \left(\frac{y}{2} - \frac{y^2}{8}\right) dy = \left[\frac{y^2}{4} - \frac{y^3}{24}\right]_0^4 = \frac{4}{3}$$

$$\bar{x} = \frac{4/3}{4/3} = 1$$

Problems

1. Find $\bar{x}$ for the area of Fig. 242, integrating first with respect to y.
2. Find $\bar{y}$ for the area of Fig. 242, integrating first with respect to x to find the moment. Check by integrating first with respect to y.

   Ans. $M = 32/15$; $\bar{y} = 1.6$.
3. Find $\bar{x}$ for the area enclosed by the line $y = 4x$ and the curve $y = x^3$. Solve two ways.

   Ans. $\bar{x} = \dfrac{64/15}{4} = 16/15$.

Example

Find $\bar{x}$ for a 60° sector of a circle of radius a by double integration, using polar coordinates.

The element of area (Fig. 243) is $dA = r\,d\theta\,dr$, and

$$x' = r\cos\theta$$

To find $\bar{x}$,

$$M = \int\int r^2 \cos\theta\,dr\,d\theta = \frac{a^3}{3}\int \cos\theta\,d\theta = \frac{a^3}{3}\Big[\sin\theta\Big]_0^{\pi/3} \qquad (1)$$

$$M = \frac{a^3\sqrt{3}}{6}$$

$$\bar{x} = \frac{M}{\pi a^2/6} = \frac{a\sqrt{3}}{\pi}$$

When r is integrated first, the first integration gives a triangular wedge extending from the center to the circumference The next integration swings this around from the horizontal axis to the line at 60°. If θ were integrated first, the area would be built up of concentric arcs. In this example, the limits of r are independent of θ, and it makes no difference which is integrated first. When the limits are not independent, it is best to integrate first with respect to r, since integration first with respect to the angle usually gives difficult forms for the second integral.

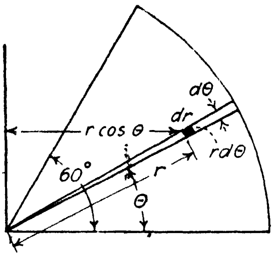

FIG. 243.

Problems

4. Find $\bar{y}$ for Fig. 243. Check from the answer of the example.
5. From the answer of the example find $\bar{x}$ and $\bar{y}$, geometrically for a 30° sector. Check by double integration.
6. Find the center of gravity of a semicircle of radius a. Memorize the result. *Ans.* $\bar{x} = 4a/3\pi$.
7. Find $\bar{x}$ and $\bar{y}$ for the area bounded by the X axis, the circle $r = a$, and the line $x = a/2$. Find the moment of the 60° sector and subtract the moment of the triangle. Divide by the difference of the areas.

118. Center of Gravity of a Solid.

When the density is constant, the center of gravity of a solid is calculated by Eq. (1) of Art. 115, which is Formula XVI with the density canceled from numerator and denominator. The element dV may be set up for single or double integration. Triple integration is seldom needed unless the density varies.

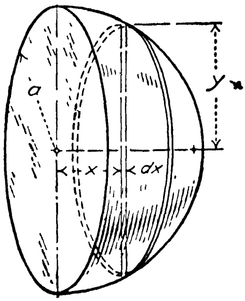

FIG. 244.

Example

Find the center of gravity of a homogeneous solid hemisphere by a single integration.

One element of volume (Fig. 244) is a circular disk of radius y and thickness dx.

$$dV = \pi y^2\,dx = \pi(a^2 - x^2)\,dx$$

$$M = \pi \int_0^a (a^2 - x^2)x\,dx = \frac{\pi a^4}{4}$$

$$x = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3a}{8} \quad \text{Memorize.}$$

Problems

1. Find the center of gravity of a segment of a homogeneous sphere of radius a which is cut off by a plane at a distance of one-half the radius from the center. *Ans.* $\bar{x} = 27a/40$.
2. Solve Prob. 1 if the plane is at one-fourth the radius from the center of the sphere.

3. **Find the distance from the vertex of a pyramid or cone to the plane through the center of gravity parallel to the base.** *Ans.* $3h/4$. **Memorize.**
4. **Find the distance of the center of gravity of a right cone from the base. Build up the cone of coaxial hollow cylinders as in Fig. 245.**
5. **A solid cylinder is cut by two planes which intersect on a diameter. One of these planes is perpendicular to the axis and the other makes an angle α with this plane, as shown in Fig. 246. Find the center of gravity of the portion of the cylinder between the planes. Solve by polar coordinates.**
Ans. $\bar{x} = 3\pi a/16$.
6. A solid cylinder of radius a is cut by two planes which make an angle α with each other and intersect on a tangent. One plane is normal to the axis of the cylinder. Find the center of gravity of the portion of the cylinder between the planes. Solve by polar coordinates with the origin of coordinates at the axis of the cylinder.

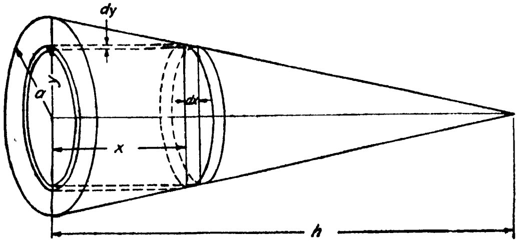

FIG. 245.

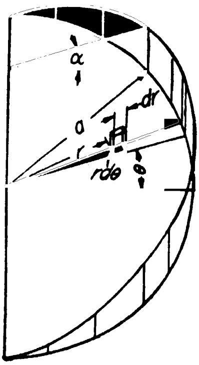

FIG. 246.

119. Miscellaneous Problems

1. Find the center of gravity of the area bounded by the X axis, the curve $y = \sin x$, and the line $x = \pi/2$. *Ans.* $\bar{x} = 1$; $\bar{y} = \pi/8 = 0.3927$.
2. Find the center of gravity of the area bounded by the X axis, the curve $y = \sin^2 x$, and the line $x = \pi/2$.
3. Find $\bar{x}$ for the area bounded by the X axis, the Y axis, the line $y = 8$, the curve $xy = 24$, and the line $x = 6$.

The total area consists of a rectangle, 3 units wide by 8 units high, and the area under the curve from $x = 3$ to $x = 6$.

$$\bar{x} = \frac{24 \times 1.5 + \int 24\,dx}{24 + \int (24/x)\,dx} = \frac{36 + 24[x]_3^6}{24 + 24\,[\log_e x]_3^6}$$

$$\bar{x} = \frac{108}{24 \times 1.69315} = 2.658$$

4. **Check Prob. 3 using the rectangle 6 units wide and 4 units high, together with horizontal elements from the Y axis to the curve.**
5. Solve Prob. 3 for $\bar{y}$ and check.
6. **Using polar coordinates, find $\bar{x}$ for the right-angled triangle bounded by the X axis, the line $x = 4$, and the line $y = \frac{4}{3}x$.**
7. **Find $\bar{x}$ for the area between the curve $y^2 = 4(x - 1)$ and the line $x = 5$.**

8. Find $\bar{y}$ for the area bounded by the curve of Prob. 7, the X axis, and the line drawn from the origin to the point (1.25,1.00).

9. Find $\bar{x}$ for the area bounded by the lines $y = 1$, $x = 2$, and the curve $y = x^3$.

10. Find $\bar{x}$ for the area between the straight line and the curve of Fig. 242 by polar coordinates. ($x = r \cos \theta$, $y = r \sin \theta$.)

11. Find $\bar{x}$ for a semicircle of radius a to the right of the Y axis with density varying as the distance from that axis. Compare with Prob. 5 of Art 118.

12. Find $\bar{x}$ for a semicircle of radius a to the right of the Y axis, with density varying as the distance from the center. What would be the form of a solid of uniform density which would have the same center of gravity?

Ans. $\bar{x} = 3a/2\pi$.

13. Find $\bar{x}$ for one-half of a vertical right cone of radius a to the right of the YZ plane. *Ans.* $\bar{x} = a/\pi$.

14. Find $\bar{x}$ for a semicircle of radius a to the right of the Y axis with density varying as the distance from the circumference.

15. The curve $y^2 = 4x$ is rotated about the Y axis to form a solid of revolution of uniform density. Find $\bar{y}$ for that part of the solid lying between the XZ plane and the plane $y = 2$.

16. The curve $y = e^x$ is rotated about the X axis. Find $\bar{x}$ for the solid of revolution so formed lying between the YZ plane and the plane $x = 2$.

17. A crucible has the form generated by revolving the curve $y = x^2$ about the Y axis with the plane $y = 1$ as the bottom. It is filled 2 ft deep (above the bottom) with molten iron weighing 480 lb per cu ft, above which floats a slag 1 ft deep weighing 160 lb per cu ft. Find the weight of the charge and locate its center of gravity.

18. Locate (in terms of the radius) the center of gravity of a thin wire bent into the form of a semicircle.

19. Show that the center of gravity of a pyramid lies on the line connecting the centroid of the base with the vertex and is one-fourth of the distance along that line from the base.

CHAPTER 13

MOMENT OF INERTIA

120. The Integrals of Mechanics. The integrals used in Chap. 12 to find the areas of surfaces, the volumes and masses of solids, and to locate center of gravity or mass centers are among a group of integrals called the *integrals of mechanics.* The integrals $\int r^2\,dm$, $\int x^2\,dA$, and $\int r^2\,dA$, which are called *moments of inertia,* and the integral $\int xy\,dA$, called the *product of inertia,* are extremely important integrals of mechanics occurring in many practical engineering problems. The *mass product of inertia,* $\int xy\,dm$, is an important integral of mechanics not treated in this chapter since most of the problems involving this integral are beyond the scope of this text. The integral $\int r^2\,dm$ involves the masses of bodies; since mass has a property known as *inertia,* its designation as moment of inertia is quite appropriate. The integrals $\int x^2\,dA$ and $\int r^2\,dA$ are functions of plane areas, which have no mass and so no inertia. These integrals are called moments of inertia because of their mathematical kinship to $\int r^2\,dm$.

121. Moment of Inertia of Solids. Every problem of dynamics which deals with rotating masses involves the moments of inertia of the rotating parts. This quantity is usually designated by the symbol I, defined as

$$I = \int r^2\,dm \quad \text{Formula XVII}$$

in which dm is the element of mass and r is the distance of the element of mass from a line, called the *axis.* It is not necessary that dm be infinitesimal in all three dimensions, but it must be infinitesimal in at least one dimension and it *must be equidistant in all parts from the axis* if Formula XVII is to be used.

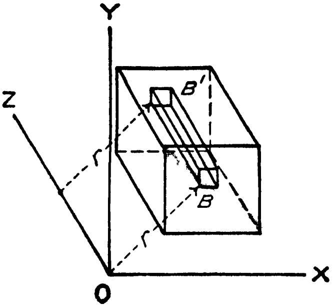

FIG. 247.

Figure 247 shows the form of the element required to find the moment of inertia with respect to the Z axis O-Z. The element B-B' is parallel to the axis O-Z. If its transverse dimensions are infinitesimal, all parts of the element are at the same distance from O-Z, within infinitesimal limits. The element may also be infinitesimal parallel to the axis or it may extend entirely through the body in this direction as shown in Fig. 247. The cross section of the element in the plane perpendicular to the axis may be rectangular as shown, or it may have the form necessary for integration with polar coordinates. The element may be a hollow cylinder of radius r and thickness dr, which extends through the body parallel to the axis.

122. Moment of Inertia by Integration. Figure 248 shows one end of a cylinder of radius a. It is desired to find the moment of inertia of this cylinder with respect to its axis, which is perpendicular to the plane of the paper.

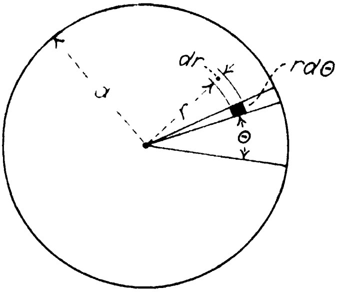

FIG. 248.

Polar coordinates are best suited for this problem. The element of volume parallel to the axis of the cylinder has the cross section $dA = r\,d\theta\,dr$. If l is the length of the cylinder, the element of volume is

$$dV = lr\,d\theta\,dr \tag{1}$$

If ρ is the density, or mass per unit volume,

$$dm = \rho lr\,d\theta\,dr \tag{2}$$

$$I = \int r^2\,dm = \rho l \int\int r^3\,dr\,d\theta \tag{3}$$

Integrating first with respect to r and putting in the limits $r = 0$ and $r = a$,

$$I = \int \frac{\rho l a^4}{4}\,d\theta \tag{4}$$

Integrating next with respect to θ and putting in the limits $\theta = 0$ and $\theta = 2\pi$,

$$I = \frac{\rho l a^4}{4}\Big[\theta\Big]_0^{2\pi} = \frac{\rho l \pi a^4}{2} \tag{5}$$

The mass of the cylinder is

$$m = \rho l \pi a^2$$

which, substituted in Eq. (5), gives

$$I = \frac{ma^2}{2} \quad \text{Formula XVIII}$$

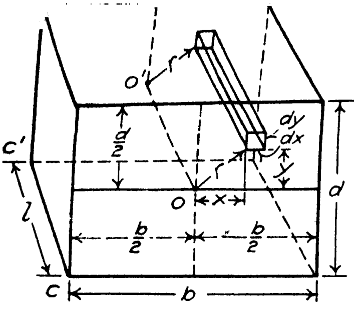

FIG. 249.

Figure 249 represents a rectangular parallelepiped of length l, breadth b, and depth d. It is desired to find its moment of inertia with respect to the axis of symmetry parallel to its length.

The line O-O' is the axis. The element of volume is $dV = l\,dx\,dy$. The element of mass is $dm = \rho l\,dx\,dy$. With O-O' as the axis of coordinates,

$$r^2 = x^2 + y^2$$

$$I = \int r^2\,dm = \rho l \int\int (x^2 + y^2)\,dx\,dy \qquad (6)$$

The limits for x are $-\frac{b}{2}$ and $+\frac{b}{2}$. The limits for y are $-\frac{d}{2}$ and $+\frac{d}{2}$.

$$I = \rho l\left(\frac{b^3d}{12} + \frac{bd^3}{12}\right) = \rho lbd\,\frac{b^2 + d^2}{12} = m\,\frac{b^2 + d^2}{12}$$

Problems

1. Find the moment of inertia of the rectangular parallelepiped of Fig. 249 with respect to the axis C-C'.

Ans. $I = \rho lbd\,\dfrac{b^2 + d^2}{3} = m\,\dfrac{b^2 + d^2}{3}$.

2. Find the moment of inertia with respect to the axis O-O' of the right-angled triangular prism of Fig. 250.

Ans. $I = \rho lbd\,\dfrac{b^2 + d^2}{12} = m\,\dfrac{b^2 + d^2}{6}$.

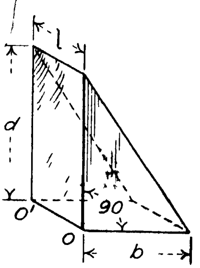

FIG. 250.

3. Calculate the moment of inertia in foot and pound units of a homogeneous solid cylinder which is 4 ft in diameter, 3 ft long, and weighs 2400 lb.

4. Calculate the moment of inertia of a homogeneous solid cylinder, which is 1 ft in diameter, 2 ft long, and weighs 480 lb per cu ft. *Ans.* $I = 30\pi = 94.248$.

5. Calculate the moment of inertia of a homogeneous hollow cylinder which is 3 ft long, has an outside diameter of 4 ft and an inside diameter of 2 ft, and weighs 40 lb per cu ft. *Ans.* $I = 900\pi = 2827.44$.

6. Find the moment of inertia of a homogeneous hollow cylinder which is 3 ft long, has an outside diameter of 4 ft and an inside diameter of 2 ft, and weighs 300 lb. *Ans.* $I = 750$.

7. Derive the equation for the moment of inertia with respect to its axis of a right circular cone. The height is h and the diameter of the base is $2a$.

Build up the cone of a series of flat disks parallel to the base, as shown in Fig. 251. The radius of each disk is r and its thickness is dy. The distance y may be measured from the base, as in Fig. 251, or from the vertex, as in Fig. 252. Since the disk is a short cylinder, its moment of inertia is given by Formula XVIII. The radius r is expressed in terms of y and constants, and the moment of inertia is derived by a single integration with y as the variable.

Ans. $I = \pi\rho h a^4/10 = 3ma^2/10$.

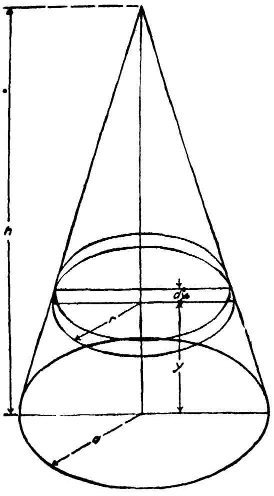

FIG. 251.

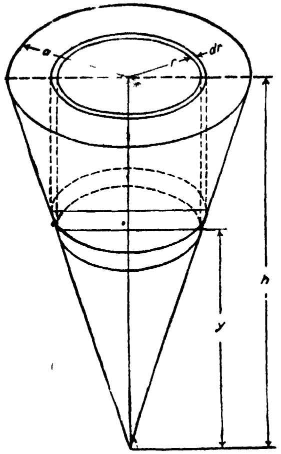

FIG. 252.

8. Solve Prob. 7 by forming the cone of coaxial hollow cylinders of radius r and thickness dr, as shown in Fig. 252.

9. Solve Prob. 7 with r as the independent variable.

The solutions of Probs. 7 and 8 are examples of the integration of elements of volume, $r\,d\theta\,dr\,dy$. Integration first with respect to θ gives a hollow ring. A second integration with respect to y builds these rings into a hollow cylinder. A third integration with respect to r expands these cylinders to form the cone. Since all parts of the hollow cylinder are at the same distance from the axis of moment of inertia, the first two integrations are unnecessary for a cone, cylinder, or sphere.

For Prob. 7 (Fig. 251), r may be integrated first and θ next, or θ may be integrated first and r next. In either case, the result is a thin disk. Since the moment of inertia of a disk has been given in Eq. (5) and Formula XVIII, it is not necessary to repeat these integrations. An integration with respect to y builds up a cone, sphere, or any other solid of revolution of a series of such disks.

10. The diameter of the base of a homogeneous right cone is 10 in. and the altitude is 12 in. Find the moment of inertia with respect to the axis of revolution, in pound and inch units, if the mass is 80 lb. *Ans.* $I = 600$.

11. Solve Prob. 10 if the weight per cubic inch is 0.3 lb. *Ans.* $I = 706.86$.
12. The diameter of the lower base of a frustum of a right cone is 12 in.; the diameter of the upper base is 8 in.; the height is 10 in.; and the weight is 38 lb. Find the moment of inertia in pound and inch units. *Ans.* $I = 506.4$.
13. Derive the equation for the moment of inertia of a sphere with respect to a diameter, using disks as elements. *Ans.* $I = 8\pi\rho a^5/15 = 2ma^2/5$.
14. Solve Prob. 13 by a single integration, using hollow cylinders as elements of volume.
15. Find the moment of inertia of a homogeneous solid cylinder, of length l and radius a, with respect to a line in its surface parallel to its axis. (One end of this cylinder is shown in Fig. 253.)
16. Find the moment of inertia of a homogeneous solid cylinder with respect to an axis parallel to the axis of the cylinder at a distance d from it.

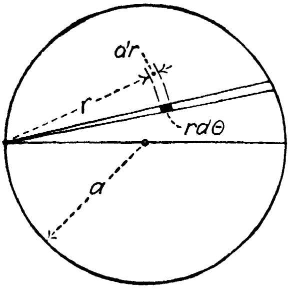

FIG. 253.

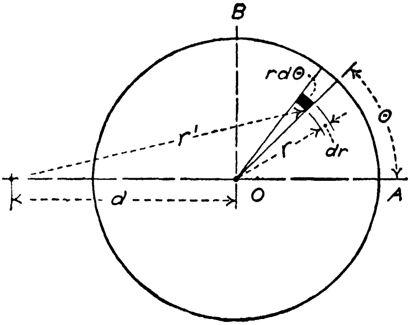

FIG. 254.

Figure 254 shows one end of the cylinder. The axis of inertia is perpendicular to the plane of the paper. To simplify the equation of the circle and the expressions for the limits of r, the origin of coordinates is placed at the axis of the cylinder instead of at the axis of inertia. The distance r' of Fig. 254 is r of Formula XVII. This distance must be expressed in terms of r, θ, and d.

Ans. $I = m[(a^2/2) + d^2]$.

123. Radius of Gyration. The radius of gyration of a body is the distance from the axis at which all the mass of the body must be placed in order that the moment of inertia may be the same as that of the actual body. It is defined mathematically by the equation

$$k^2 = \frac{I}{m} \qquad \text{Formula XIX}$$

in which k is the radius of gyration.

The moment of inertia of a homogeneous solid cylinder with respect to its axis is $ma^2/2$, and $k^2 = a^2/2$. If a homogeneous solid cylinder of given mass is replaced by a thin hollow cylindrical shell of the same mass, the moment of inertia of the shell will equal that of the cylinder, provided the radius of the shell is 0.7071 of the radius of the cylinder.

Problems

1. Calculate the radius of gyration of a homogeneous solid sphere 10 in. in diameter. *Ans.* $k = \sqrt{10} = 3.16$ in.
2. Derive the expression for the radius of gyration of a homogeneous solid sphere of radius r. *Ans.* $k = 0.63246r$.
3. Derive the expression for the radius of gyration of a homogeneous solid cylinder of radius r. *Ans.* $k = 0.7071r$.
4. Derive the expression for the square of the radius of gyration of a hollow cylinder, of which the outside radius is a and the inside radius is b. *Ans.* $k^2 = (a^2 + b^2)/2$.
5. Derive the expression for the square of the radius of gyration of a homogeneous solid cone of radius a with respect to the axis of the cone. *Ans.* $k^2 = 3a^2/10$.
6. Find the radius of gyration of a homogeneous hollow cylinder, of which the outside diameter is 16 in. and the inside diameter is 10 in. *Ans.* $k = 6.67$ in.
7. Find the radius of gyration of a homogeneous hollow cylinder, of which the outside diameter is 16 in. and the inside diameter is 15 in. *Ans.* $k = 7.75$ in.

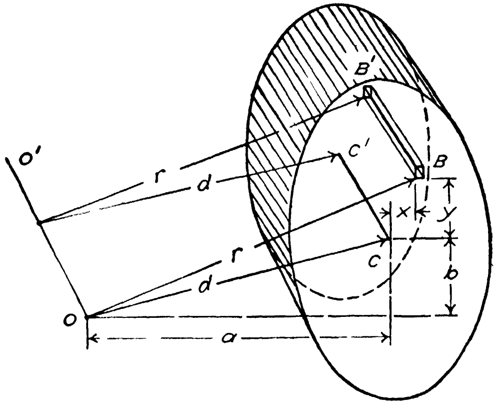

FIG. 255.

124. Transfer of Axis. The calculation of the moment of inertia about any axis is frequently a laborious process. The work may often be simplified by means of a transfer of the axis. If the moment of inertia of a body is known with respect to some axis, its moment of inertia with respect to any other axis parallel to that axis may be found by simple algebraic methods.

In Fig. 255, the moment of inertia is desired with respect to O-O'. Let B-B' represent any element of the body parallel to O-O'. The distance from O-O' to B-B' is r.

$$I = \int r^2\,dm \tag{1}$$

Suppose that the moment of inertia of the body is known with respect to the axis C-C', which is parallel to the axis O-O' at a distance d. The horizontal component of d is a and the vertical component is b.

$$d^2 = a^2 + b^2 \tag{2}$$

The coordinates of the element B-B' with respect to C-C' as the Z axis are x and y, and the coordinates with respect to O-O' are $a + x$ and $b + y$

$$r^2 = (a + x)^2 + (b + y)^2 \tag{3}$$

$$I = \int (a^2 + 2ax + x^2 + b^2 + 2by + y^2)\, dm \tag{4}$$

$$I = \int (a^2 + b^2)\, dm + \int (x^2 + y^2)\, dm + 2a\int x\, dm + 2b\int y\, dm \tag{5}$$

The first term of the second member of Eq. (5) is

$$(a^2 + b^2)\int dm = d^2\int dm = md^2 \tag{6}$$

The second term is the moment of inertia with respect to the axis C-C', which may be represented by I_c.

$$\int (x^2 + y^2)\, dm = I_c \tag{7}$$

The third term is $2a\int x\, dm$. The expression $x\, dm$ is the moment of the element B-B' of mass dm with respect to the vertical plane through C-C'. The expression $\int x\, dm$ is the moment of the entire volume with respect to this plane. If the *vertical plane* through C-C' passes through the center of gravity of the body, the total moment is zero. Similarly, $\int y\, dm$ is equal to zero if the *horizontal plane* through C-C' passes through the center of gravity. If the *line* C-C' passes through the center of gravity of the body, the last two terms of Eq. (5) vanish and

$$I = I_c + md^2 \qquad \text{Formula XX}$$

in which I_c is the moment of inertia with respect to an axis through the center of gravity, I is the moment of inertia with respect to a parallel axis at a distance d from the center of gravity, and m is the mass of the body.

Formula XX affords a convenient method of finding the moment of inertia of a body with respect to any axis, if the moment of inertia is known with respect to a parallel axis through the center of gravity. If the moment of inertia is known for some axis which does not pass through the center of gravity, and if it is desired for some other axis which does not pass through the center of gravity, Formula XX may be used to transfer from the first axis to the center of gravity and then to transfer from the center of gravity to the second axis.

If k_c is the radius of gyration of the body with respect to the axis C-C' and k is the radius of gyration with respect to the axis O-O',

$$k^2 = \frac{I}{m} = \frac{mk_c^2 + md^2}{m} = k_c^2 + d^2 \qquad \text{Formula XXI}$$

The moment of inertia with respect to axes through the center of gravity is given in handbooks for many of the common solids. By means of Formula XX and the handbook, the engineer can quickly calculate the moment of inertia of any solid for an axis parallel to the axis for which I_c is known. Sometimes the moment of inertia is known for an axis which does not pass through the center of gravity. To find the moment of inertia with respect to a parallel axis, transfer first to an axis through the center of gravity to get I_c and then transfer to the required axis.

Problems

1. A homogeneous solid cylinder is 8 in. in diameter, 12 in. long, and weighs 50 lb. Find the moment of inertia with respect to an axis which is 15 in. from the axis of the cylinder. *Ans.* $I = 11{,}650$.
2. Find the radius of gyration of the cylinder of Prob. 1 with respect to the given axis. *Ans.* $k^2 = 233$; $k = 15.264$ in.
3. A solid sphere is 10 in. in diameter and weighs 40 lb. Find the radius of gyration with respect to an axis tangent to the surface. *Ans.* $k^2 = 35$; $k = 5.916$ in.
4. Find the radius of gyration of a hollow cylinder, 2 in. inside diameter, 10 in. outside diameter with respect to a line 5 in. from its axis. *Ans.* $k^2 = 38$; $k = 6.164$ in.
5. Solve Prob. 1 of Art. 122 by means of Formula XX.
6. Derive the expression for the moment of inertia of a homogeneous solid hemisphere with respect to an axis through its center of gravity parallel to the plane which bounds it. *Ans.* $I_c = 83ma^2/320$.
7. A circular disk, 20 in. in diameter, has four holes drilled through it. The diameter of each hole is 5 in. and the center is 4 in. from the center of the disk. Find the radius of gyration. *Ans.* $k = 7.76$ in.

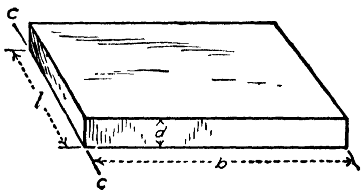

FIG. 256.

125. Moment of Inertia of a Thin Plate. The moment of inertia of a rectangular parallelepiped of length l, breadth b, and thickness d, with respect to one edge parallel to its length (Fig. 256) is

$$I = \rho lbd\frac{b^2 + d^2}{3} = m\frac{b^2 + d^2}{3}$$

(See Prob. 1 of Art. 122.) If the thickness is small compared with the breadth, its square may be relatively negligible and may, therefore, be neglected. For instance, if $b = 100$ in. and $d = 1$ in., the error made by dropping d^2 is only 1 part in 10,000. The moment of inertia of a long slender rod with respect to an axis through one end perpendicular to its length is $ml^2/3$, and the moment of inertia with respect to an axis through the middle perpendicular to its length is $ml^2/12$. These formulas are derived from the results for a parallelepiped by neglecting the thickness and interchanging l and b.

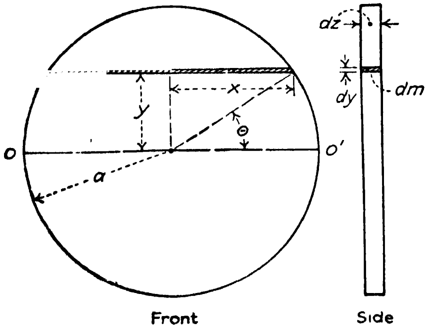

FIG. 257.

Problems

1. Find the moment of inertia of a thin circular disk of thickness dz with respect to a diameter.

If the horizontal diameter, O-O' (Fig. 257), is taken as the axis of reference, the element of volume parallel to this diameter is $2x\,dy\,dz$. The thickness dz is so small (many times less than shown in the side view of Fig. 257) that its square may be neglected in calculating r.

$$r = y$$

$$I = \rho\,dz \int 2xy^2\,dy$$

which is best solved by substituting $x = a\cos\theta$, $y = a\sin\theta$.

$$I = 2\rho\,dz\,a^4 \int \cos^2\theta\sin^2\theta\,d\theta = \rho a^4\,dz \int \frac{\sin^2 2\theta}{2}\,d\theta$$

$$I = \rho a^4\,dz \int \frac{1 - \cos 4\theta}{4}\,d\theta = \rho a^4\,dz \left[\frac{\theta}{4} - \frac{\sin 4\theta}{16}\right]_{-\pi/2}^{\pi/2}$$

$$I = \rho\,dz\frac{\pi a^4}{4} = \frac{ma^2}{4}$$

This moment of inertia is one-half that of the disk with respect to its axis.

2. Find the moment of inertia of a rectangular plate of breadth b, length l, and thickness dz, with respect to an axis through its center parallel to its length.

Ans. $I = \rho l dz(b^3/12) = mb^2/12$.

3. Find the moment of inertia of a triangular plate of height h, base b, and thickness dz, with respect to an axis through its center of gravity parallel to its base. Find the moment of inertia first with respect to an axis through the vertex parallel to the base. Transfer to a parallel axis through the center of gravity by means of Formula XX. *Ans.* $I = \rho b\,dz(h^3/36) = mh^2/18$.

126. Plate Elements. The moment of inertia of a solid can often be found conveniently by a single integration when the moments of inertia of thin plates with respect to their gravity axes are known. A suitable element in the form of a thin plate of infinitesimal thickness with its plane parallel to the axis of the desired moment of inertia is chosen. Then the moment of inertia of the elemental plate with respect to the desired axis is set up by means of Formula XX, and the integration of the moments of inertia of the elements into that of the entire solid is carried out. The process is described mathematically by the expression

$$I = \int dI \qquad \text{Formula XXII}$$

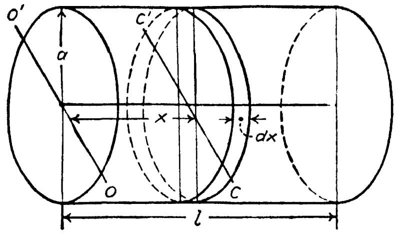

FIG. 258.

Example

Find the moment of inertia of a homogeneous solid cylinder of length l and radius a with respect to a diameter at one end (Fig. 258).

From Prob. 1 of Art. 125 the moment of inertia of a disk of radius a and thickness dx with respect to the diameter C-C' is

$$dI_c = \rho \frac{\pi a^4}{4} dx \tag{1}$$

The moment of inertia with respect to the parallel axis O-O' at a distance x from C-C' is

$$I = \int dI = \int (dI_c + x^2\, dm) = \rho\pi \frac{a^4}{4} \int dx + \rho\pi a^2 \int x^2\, dx \tag{2}$$

$$I = \rho\pi \left[\frac{a^4x}{4} + \frac{a^2x^3}{3}\right]_0^l = \rho\pi l a^2 \left(\frac{a^2}{4} + \frac{l^2}{3}\right) \tag{3}$$

$$I = m\left(\frac{a^2}{4} + \frac{l^2}{3}\right) \tag{4}$$

The first term of Eq. (4) is called *the moment of inertia with respect to the plane of the base*, and the second term is *the moment of inertia with respect to a diametral plane of the cylinder*. The moment of inertia of a solid with respect to a plane has no direct usefulness but is a convenient concept in the study of moments of inertia. If the moments of inertia of a solid with respect to two planes at right angles to each other are added, the sum is the moment of inertia with respect to the line of intersection, which is a useful quantity in mechanics. If the moment of inertia of a third plane at right angles to the other two is added to the moment of inertia with respect to a line, the total is *the moment of inertia with respect to the point where the three planes intersect.* Note that the second term of Eq. (4) would be the moment of inertia of the cylinder if it were compressed into a flat plate at the bottom of the cylinder and the first term (see Prob. 2 below) would be the moment of inertia of the cylinder with respect to 0-0′ if the cylinder were condensed into the line of its axis.

Problems

1. Find the moment of inertia of a homogeneous solid cylinder with respect to a diameter at the middle.

Ans. $I = m\left(\frac{l^2}{12} + \frac{a^2}{4}\right)$.

2. Find the moment of inertia of a circular rod with respect to an axis through one end perpendicular to its length, if the diameter is so small that its square is negligible compared with the square of the length. *Ans.* $I = ml^2/3$.

3. Find the moment of inertia of a homogeneous right cone of altitude h and diameter $2a$ with respect to an axis through the vertex parallel to the base.

Ans. $I = \frac{3m}{5}\left(\frac{a^2}{4} + h^2\right)$.

4. Calculate the radius of gyration of a right cone 12 in. in diameter at the base and 18 in. high with respect to an axis through the vertex parallel to the base.

Ans. $k = 14.135$ in.

5. Derive the expression for the moment of inertia of a homogeneous right cone with respect to an axis through the center of gravity parallel to the base and also with respect to a diameter of the base.

Ans. $I_c = \frac{3m}{5}\left(\frac{a^2}{4} + \frac{h^2}{16}\right)$; $I_b = \frac{3m}{5}\left(\frac{a^2}{4} + \frac{h^2}{6}\right)$.

6. Find the radius of gyration of a flywheel with hub 2 in. inside diameter and 4 in. outside diameter, 6 in. wide; with rim 20 in. inside diameter and 24 in. outside diameter, 4 in. wide; and with six spokes 2 in. in diameter, considering all parts as cylinders and neglecting the fillets. (Assume the spokes 18 in. long with one end 2 in. from the center of the wheel.)

7. Find the moment of inertia of a sphere about a diameter, using plate elements parallel to the diameter which is the axis of moment of inertia.

8. Find the moment of inertia with respect to the X axis of the solid formed by revolving the curve $y = x^2$ about the Y axis and cutting off the upper portion by the plane $y = 3$.

127. Moment of Inertia of a Plane Area. Many problems in engineering involve the resultants of pressures distributed over finite areas and varying in intensity directly with the distance from some line or point. For example, the pressure of fresh water (at about 50°F) on a gate or dam is at every point $62.4y$ pounds per square foot, where y is the vertical depth of submergence of the point below the free surface of the water. The total pressure on a plane area is then $P = \int 62.4y\,dA$, dA being an element of the area, and $P = 62.4\bar{y}A$, where $\bar{y}$ is the vertical depth of submergence of the centroid of the area. But the *location* of this resultant force *is not at the centroid of the area* (except when the area is horizontal) but at some other point whose position must be determined by moments. A convenient axis of moments is the line where the area extended, if necessary, cuts the free surface of the water (extended if necessary). The element of force $62.4y\,dA$ has a moment about this line equal to $(y \csc \theta)(62.4y\,dA)$, where θ is the angle between the area under consideration and the surface of the water. The total moment is then $M = 62.4 \csc \theta \int y^2\,dA$. Instead of solving such problems by evaluating the definite integrals involved, it is better separately to study the property of the plane area represented by the integral and called the *moment of inertia* of the area. Usually the moment of inertia can be found without the necessity of integrating.

If a vertical plane passed through the centroid of the area under pressure, discussed above, does not divide the area into two symmetrical parts, the resultant pressure may not lie in this vertical plane, and its lateral location must be found by moments. The intersection of this vertical plane and the area in question makes a convenient axis of moments and the moment of the distributed pressure about this axis is $62.4\int xy\,dA$, where x is the distance of the element from this axis. The integral $\int xy\,dA$ is called the *product of inertia.* It has the same dimensions as does the moment of inertia and, consequently, can be added to the moment of inertia without violating the rules of mathematical propriety. These quantities are both scalar with the dimension of L^4 and neither can be expected to make any appeal to common sense. In addition to the example given above, nearly all the problems of beams, columns, torsion members, etc., in strength of materials involve these integrals. They get their names from the mathematical similarity to mass moments and products of inertia, the sole difference being that an element of mass is replaced by an element of area. Moment of inertia of a plane area is defined mathematically as

$$I = \int r^2\,dA$$

and as in the case of mass moment of inertia, $I = \int r^2\,dm$, the element of area must be at all parts equidistant from the axis of I. The same symbol, I, is usually used for both since both kinds of moment of inertia rarely appear in the same problem.

The radius of gyration of a plane area is defined by the equation

$$k^2 = \frac{\text{moment of inertia}}{\text{area}} \tag{1}$$

The formula for the transfer of axis for the moment of inertia of a plane area is

$$I = I_c + Ad^2 \tag{2}$$

in which I_c is the moment of inertia with respect to an axis through the center of gravity of the area, I is the moment of inertia with respect to any parallel axis, and d is the distance between these axes.

128. Polar Moment of Inertia. When the axis is normal to the plane of the area, $\int r^2\,dA$ is called the *polar moment of inertia* of the area. The polar moment of inertia of a plane area is equivalent to the moment of inertia of a right prism or cylinder for an axis parallel to its length, provided the density and length are such that the mass per unit area is unity.

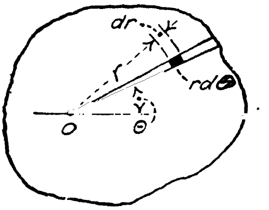

Fig. 259.

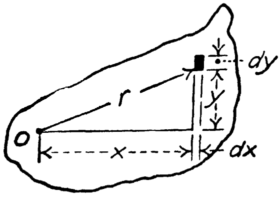

Fig. 260.

It is customary to represent the polar moment of inertia by J.

$$J = \int r^2\,dA \qquad \text{Formula XXIII}$$

In Fig. 259, O represents an axis perpendicular to the plane of the paper. The element of area is $r\,d\theta\,dr$ and

$$J = \int\int r^3\,d\theta\,dr \qquad (1)$$

In Fig. 260, the element of area is $dx\,dy$ and

$$J = \int r^2\,dA = \int\int (x^2 + y^2)\,dx\,dy \qquad (2)$$

Problems

1. Find the polar moment of inertia of a circle of radius a with respect to an axis through its center and compare the moment of inertia and radius of gyration with those of a cylinder. *Ans.* $J = \pi a^4/2 = Aa^2/2;\ k^2 = a^2/2.$
2. Find the polar moment of inertia of a rectangle of breadth b and depth d with respect to an axis through its center.
$$\textit{Ans. } J = \frac{bd^3 + b^3d}{12} = \frac{A(b^2 + d^2)}{12};\ k^2 = \frac{b^2 + d^2}{12}.$$
3. Find the polar moment of inertia of an isosceles triangle of base b and altitude h with respect to an axis through the vertex.
$$\textit{Ans. } J = \frac{bh^3}{4} + \frac{b^3h}{48}.$$

129. Axis in a Plane. When $\int r^2\,dA$ is taken with respect to an axis in the plane of the area, it is called simply the *moment of inertia* of the area. This moment of inertia is a factor in the strength and stiffness of beams and columns and has much greater application than the polar moment of inertia. The values of this moment of inertia for the common geometrical figures and for sections of the ordinary structural shapes are given in the handbooks of the steel companies.

If the moment of inertia is taken with respect to the X axis, $r = y$, and

$$I = \int y^2\, dA = I_x \tag{1}$$

If the moment of inertia is taken with respect to the Y axis, $r = x$, and

$$I = \int x^2\, dA = I_y \tag{2}$$

The moment of inertia with respect to the X axis may be designated by I_x and the moment of inertia with respect to the Y axis by I_y. This convention is convenient, but it is not in general use.

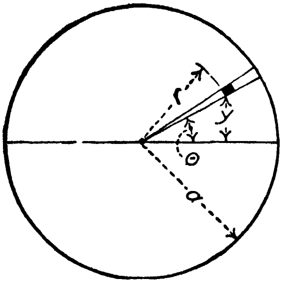

FIG. 261.

Example

Find the moment of inertia of a circular area of radius a with respect to a diameter.

Figure 261 applies to this example. The moment of inertia is taken with respect to the X axis and the element of area is given in polar coordinates.

$$I = \int y^2\, dA = \int\int r^3\, dr \sin^2\theta\, d\theta = \left[\frac{r^4}{4}\right]_0^a \int \sin^2\theta\, d\theta \tag{1}$$

$$I = \frac{a^4}{4}\int \sin^2\theta\, d\theta = \frac{a^4}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{\pi a^4}{4} = \frac{Aa^2}{4} \tag{2}$$

Problems

1. Find the radius of gyration of a circular area with respect to a diameter. *Ans.* $k = a/2$.
2. Find the moment of inertia of a rectangle of base b and altitude d with respect to a line through the center parallel to the base. *Ans.* $I = bd^3/12$.
3. Find the moment of inertia of the rectangle of Prob. 2 with respect to the base. Solve by integration and also by transfer of axis. *Ans.* $I = bd^3/3$.
4. The base of a rectangle is 6 in. and its altitude is 10 in. Find its moment of inertia with respect to an axis 4 in. below the base and parallel to it. Solve two ways.

$$Ans.\ I = \frac{6 \times 1000}{12} + 60 \times 81 = 5360 \text{ in.}^4$$

$$I = \frac{6 \times 2744}{3} - \frac{6 \times 64}{3} = 5360 \text{ in.}^4$$

5. Derive the expression for the moment of inertia of a triangle of base b and altitude h with respect to an axis through the vertex parallel to the base.
Ans. $I = bh^3/4$.

6. Using the answer of Prob. 5, find the moment of inertia of a triangle with respect to an axis through the center of gravity parallel to the base, and find the moment of inertia with respect to the base.
Ans. $I_c = bh^3/36$; $I = bh^3/12$.

7. If b is the length of each of two opposite sides of a parallelogram and h is the perpendicular distance between them, find the moment of inertia of the parallelogram with respect to one of these sides. *Ans.* $I_b = bh^3/3$.

8. A trapezoid has the lower base 8 in., the upper base 2 in., and the altitude 6 in. Find the moment of inertia with respect to the lower base by dividing the area into a parallelogram and a triangle. *Ans.* $I = 252$ in.^{4}

9. Solve Prob. 8 by dividing the area into two triangles.

10. Find the moment of inertia of the trapezoid of Prob. 8 with respect to an axis through the center of gravity parallel to the base. Solve by transfer from the base.
Ans. $I_c = 252 - 172.8 = 79.2$ in.

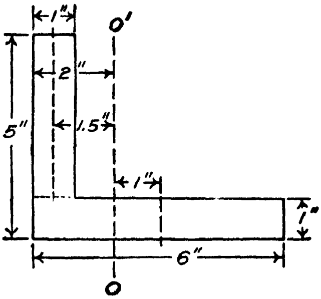

FIG. 262.

11. Find the moment of inertia of a 6- by 5- by 1-in. angle section with respect to an axis through the center of gravity parallel to the 5-in. leg (Fig. 262). The center of gravity is found to be 2 in. from the back of the 5-in. leg. To find the moment of inertia with respect to the axis O-O', the section is divided into a 6- by 1-in. rectangle and a 1- by 4-in. rectangle. The moment of inertia of each rectangle is first found with respect to a vertical line through its center of gravity and then transferred to the axis O-O'.

1- by 4-in. rectangle

$$\frac{4 \times 1^3}{12} + 4\left(\frac{3}{2}\right)^2 = \frac{1}{3} + 9 = 9\frac{1}{3}$$

6- by 1-in. rectangle

$$\frac{1 \times 6^3}{12} + 6 \times 1^2 = 18 + 6 = 24$$

Total $I_c = 33\frac{1}{3}$

The problem may also be solved by finding first the moment of inertia of the entire section with respect to some vertical axis and then by transferring to the parallel axis through the center of gravity. The moment of inertia with respect to the back of the 5-in. leg is

$$I = \frac{1 \times 6^3}{3} + \frac{4 \times 1^3}{3} = 73\frac{1}{3}$$

This axis is 2 in. from the axis O-O'.

$$I_c = 73\tfrac{1}{3} - 10 \times 2^2 = 33\tfrac{1}{3}$$

12. Solve Prob. 11 by dividing the section into two equal rectangles as shown in Fig. 263. Find I with respect to the axis C-C'' and then transfer to the axis O-O' through the centroid. Solve also by transfer *from* the centroid of each rectangle as in the first method of Prob. 11.

13. Find the moment of inertia of the 6- by 5- by 1-in. angle section of Figs. 262 and 263 with respect to the axis through the center of gravity parallel to the 6-in. leg. Solve by two methods. Calculate the radius of gyration and compare the results with "Carnegie Pocket Companion" or the "American Institute of Steel Construction Handbook" for a similar smaller section. *Ans.* $I = 20\tfrac{5}{8}$.

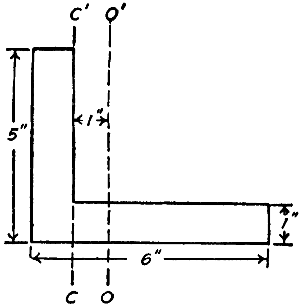

FIG. 263.

14. Figure 264 shows a standard channel section without fillets or curves. The slope of the flange of the standard section is $\tfrac{1}{6}$. Find the moment of inertia of this section with respect to an axis through the center of gravity perpendicular to the web.

$$Ans.\ I = \frac{bd^3}{12} - \frac{h^4 - l^4}{16}.$$

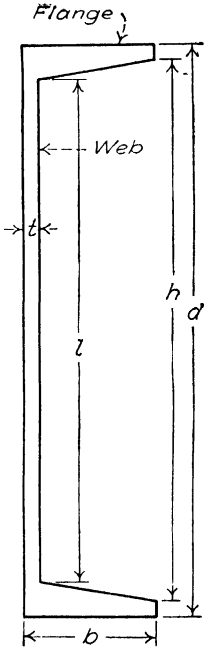

FIG. 264.

15. Look up the dimensions of a 15-in. 33.9-lb standard channel section. Calculate the moment of inertia with respect to an axis which is perpendicular to the web and passes through the center of gravity by means of the answer to Prob. 14. Calculate the radius of gyration and compare with the handbook.

16. Find the moment of inertia of the channel section of Prob. 15 with respect to an axis along the right side of the web of Fig. 264. Find the distance of the center of gravity from this axis and calculate the moment of inertia with respect to the axis through the center of gravity parallel to the web. Check with your handbook.

17. A 12- by 1-in. plate is riveted to the lower flange of a 10-in. 30-lb I beam. How far is the center of gravity of the combination from the bottom of the beam? Find the moment of inertia with respect to the axis through the center of gravity parallel to the flange. Get all the data you can from the handbook. Check your solution.

18. A plane area is bounded by the X axis, the curve $y^2 = 4x$, and the line $x = 9$. Find the moment of inertia with respect to the line $x = -2$.

19. Find the moment of inertia of the area bounded by the curve $xy = 24$, the line $x = 2$, and the line $y = 3$, with respect to the X axis.

20. Find the moment of inertia of the area bounded by the curve $y = x^2$, the line $x = 4$, and the X axis with respect to the line $x = -2$.

130. Product of Inertia. The product of inertia is represented by H in algebraic equations.

$$H = \int xy\,dA \tag{1}$$

The product of inertia has no physical significance. It is, however, so useful in the calculation of the moment of inertia of plane areas that it is desirable to master some of its properties.

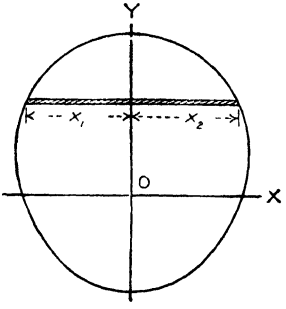

Fig. 265.

If an area has an axis of symmetry, the product of inertia with respect to that axis and the axis perpendicular to it is zero. In Fig. 265, the Y axis is an axis of symmetry. Any horizontal line extends equal distances on each side of the Y axis. If Eq. (1) is integrated first with respect to x,

$$H = \int xy\,dA = \int\int xy\,dy\,dx = \int \tfrac{1}{2}[x^2]_{x_1}^{x_2} y\,dy = \tfrac{1}{2}\int [x_2^2 - x_1^2]y\,dy$$

When the area is symmetrical with respect to the Y axis, x_1 is numerically equal to x_2 but has the opposite sign. Since the square of a negative quantity is positive, x_1^2 has the same sign as x_2^2, and

$$x_2^2 - x_1^2 = 0$$

Consequently,

$$H = 0 \tag{2}$$

Problems

1. By integration find the product of inertia of a rectangle of width b and height d with respect to the lower edge and the left edge as axes. *Ans.* $H = b^2d^2/4$.

2. Solve Prob. 1 with respect to the lower edge and the right edge as axes. *Ans.* $H = -b^2d^2/4$.

3. Find the product of inertia of the first quadrant of a circle of radius a with respect to the X and Y axes. *Ans.* $H = a^4/8$.

4. Calculate the product of inertia of a rectangle 10 in. wide and 6 in. high with respect to the lower 10-in. edge and a vertical line 3 in. from the left 6-in. edge as axes. Solve first by integration and check by means of the answers of Probs. 1 and 2.

5. Calculate the product of inertia with respect to the coordinate axes of the area bounded by the X axis, the line $4x = 3y$, and the line $x = 6$. *Ans.* $H = 288$ in.^{4}

6. Calculate the product of inertia of the area of Prob. 18 (Art. 129) with respect to the coordinate axes (first quadrant). *Ans.* $H = 486$ in.^{4}

7. Calculate the product of inertia with respect to the coordinate axes of the area bounded by the Y axis, the line $y = 6$, and the curve $y^2 = 4x$.

8. Calculate the product of inertia with respect to the coordinate axes of the area bounded by these axes and the line whose intercepts are $x = 6$ and $y = 4$. *Ans.* $H = 24$ in.^{4}

9. Find the product of inertia with respect to the coordinate axes of the area bounded by these axes and the line whose intercepts are $(0,d)$ and $(b,0)$. *Ans.* $H = b^2d^2/24$.

131. Transfer of Axes for Product of Inertia. In Fig. 266, O-X and O-Y are a pair of axes at right angles to each other, and O'-X' and O'-Y' are a second pair of axes parallel to these. The coordinates of the point O with respect to the axes O'-X' and O'-Y' are b and a, respectively. It is desired to find the product of inertia with respect to O'-X' and O'-Y' in terms of the product of inertia with respect to the first pair of axes and the distances a and b.

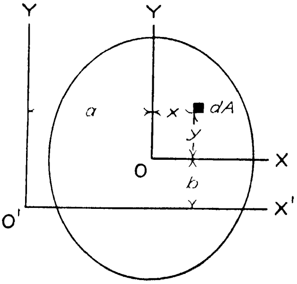

FIG. 266.

The coordinates of the element dA with respect to O-X and O-Y are y and x. The coordinates with respect to O'-X' and O'-Y' are $b + y$ and $a + x$. The product of inertia is given by the equation

$$H = \int (a + x)(b + y)\, dA \tag{1}$$

$$H = ab \int dA + a \int y\, dA + b \int x\, dA + \int xy\, da \tag{2}$$

The first term of the second member of Eq. (2), $ab\int dA$, is equivalent to abA. The last term, $\int xy\, dA$, is the product of inertia with respect to the axes O-X and O-Y, which may be written H_c. The expression $\int y\, dA$ is the total moment of the area with respect to the axis O-X. If this axis passes through the center of gravity of the area, the moment of the area is zero and the second term vanishes. The expression $\int x\, dA$ represents the moment of the area with respect to the axis O-Y. If the axis O-Y passes through the center of gravity, the third term also vanishes. If the point O is at the center of gravity of the area

$$H = H_c + abA \qquad \text{Formula XXIV}$$

in which H_c is the product of inertia with respect to a pair of axes which intersect at the center of gravity. If either O-X or O-Y is an axis of symmetry, H_c also becomes zero and Formula XXIV reduces to

$$H = abA$$

If the point O lies in the first quadrant with respect to the axes O'-X' and O'-Y', both a and b are positive and the product $a \times b$ is positive. If O lies in the third quadrant, both a and b are negative and their product is positive. If O lies in the second or fourth quadrant, the product $a \times b$ is negative. The product of inertia with respect to the axes through the center of gravity may be positive, negative, or zero. While moment of inertia is always positive, product of inertia may be either positive or negative.

Problems

1. By means of Formula XXIV calculate the product of inertia of a rectangle 8 in. wide and 6 in high with respect to the lower and left edges as axes.
   Ans. $H = 0 + 4 \times 3 \times 48 = 576$ in.^{4}
2. Calculate the product of inertia of the area of Prob. 1 with respect to the lower edge and a line 1 in to the left of the right edge as axes.
   Ans. $H = -432$ in.^{4}
3. Find the product of inertia of the area of Prob. 5 (Art. 130) with respect to horizontal and vertical axes through the center of gravity by means of Formula XXIV.
   Ans. $H_c = 32$ in.^{4}
4. Find the product of inertia of a 6- by 5- by 1-in. angle section in the position of Fig. 267 with respect to the backs of the legs as axes.

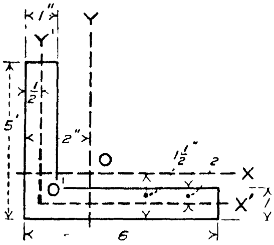

FIG. 267.

5. Find the product of inertia of the angle section of Prob. 4 with respect to axes through the center of gravity parallel to the legs.
   The section may be divided into two rectangles. The product of inertia may be transferred from lines through the center of gravity of each of these rectangles to the parallel lines through the center of gravity of the figure.
   Another way is to find first the product of inertia with respect to the axes O'-X' and O'-Y' which bisect the legs. The product of inertia for these axes is zero, since O'-X' is an axis of symmetry for the horizontal leg and O'-Y' is an axis of symmetry for the vertical leg. Transferring to the center of gravity,

$$0 = H_c + abA = H_c + 1.5 \times 10$$
$$H_c = -15 \text{ in.}^4$$

6. Find the product of inertia of a semicircle in the first and fourth quadrants with respect to the diameter which bounds it and the tangent at the lower end of this diameter.

132. Algebraic Sign of Product of Inertia. The algebraic sign of the product of inertia of an area has no physical significance since it is merely a function of the orientation of the area with respect to the axes, *For example, if the angle section of Prob. 5 of Art. 131 is rotated 180° about either the X or the Y axis, the sign of H is reversed and the answer to the problem is +15 in.*^{4} But this algebraic sign must be carefully determined and used as a significant factor in equations involving H, and the orientation of the area originally set up must be adhered to throughout the solution. It is obvious that areas lying in the first and third quadrants have positive H while those in the other two quadrants have negative H. Product of inertia refers always to a *pair* of axes (usually at right angles to each other), so that the answer to Prob. 5 of Art. 131 should be stated, logically, as $H_{xy} = -15$. It is convenient and customary, however, to state $H_x = -15$, the pair of axes being understood and the first quadrant being understood as the quadrant entered by a positive rotation of the positive end of the X axis. It is also convenient to state, for the same problem (and the same answer really), $H_y = +15$, the pair of rectangular axes being understood, as before, and the first quadrant being understood as the quadrant entered by rotating positively the positive end of the Y axis. Then for any pair of rectangular axes, any area has H_x and H_y numerically equal but of opposite sign. $H_x = -H_y$. This is the notation as used in Art. 135.

133. Rotation of Axes. In Fig. 268, O-X and O-Y are a pair of axes for which the moments of inertia are known or may be easily calculated. These are $I_x = \int y^2\,dA$ and $I_y = \int x^2\,dA$. It is desired to find the moment of inertia with respect to the axis O-X' which makes an angle θ with O-X.

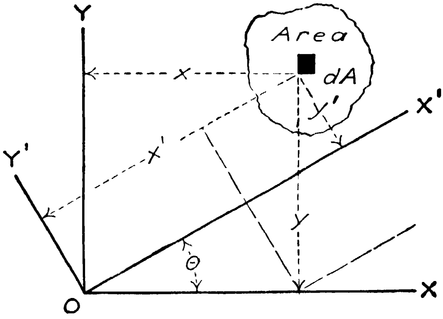

FIG. 268.

Let O-Y' be an axis perpendicular to O-X'. The coordinates of the element dA with respect to these axes are (x',y') of Fig. 268. The moment of inertia with respect to O-X' is

$$I' = \int y'^2\,dA \tag{1}$$

From the geometry of the figure,

$$y' = y\cos\theta - x\sin\theta \tag{2}$$

$$I' = \int (y^2\cos^2\theta - 2xy\cos\theta\sin\theta + x^2\sin^2\theta)\,dA \tag{3}$$

$$I' = I_x\cos^2\theta + I_y\sin^2\theta - 2\cos\theta\sin\theta\int xy\,dA \tag{4}$$

$$I' = I_x\cos^2\theta + I_y\sin^2\theta - H\sin 2\theta \tag{5}$$

$$I' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - H\sin 2\theta \tag{6}$$

When the angles are such that the square of the sine and cosine are easily calculated, Eq. (5) may be used. Equation (6) is preferable for most cases. To find the direction of the axis through any given point for which the moment of inertia is greater than for any other axis through that point, the mathematical conditions for maximum and minimum are applied to Eq. (6).

$$I' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - H\sin 2\theta \tag{7}$$

Differentiating with respect to θ,

$$\frac{dI}{d\theta} = (I_y - I_x)\sin 2\theta - 2H\cos 2\theta \tag{8}$$

Equating the derivative to zero and solving for θ,

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{9}$$

From Fig. 268, the product of inertia with respect to the axes O-X' and O-Y' is

$$H' = \int x'y'\,dA \tag{10}$$

$$H' = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA \tag{11}$$

$$H' = (\cos^2\theta - \sin^2\theta)\int xy\,dA + \cos\theta\sin\theta\int (y^2 - x^2)\,dA \tag{12}$$

$$H' = H\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta \tag{13}$$

If H' is zero, the second member of Eq. (13) is zero, and

$$H\cos 2\theta = \frac{I_y - I_x}{2}\sin 2\theta \tag{14}$$

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{15}$$

Since the tangent of an angle may have any value from minus infinity to plus infinity, there is always some direction for which the product of inertia for a pair of axes through any point is zero. If there is a line of symmetry through the point, this line is one of the axes for which the product of inertia is zero.

Equation (15) is identical with Eq. (9), *i.e.*, *the axes of zero product of inertia are the axes of maximum and minimum moments of inertia.* Such a pair of axes passing through the centroid of a plane area are called the *principal axes* of the area. These are the axes of greatest importance in problems of strength of materials.

Problems

1. Find the moment of inertia of a 6- by 5-in. rectangle with respect to an axis through the center at an angle of 30° with the 6″ leg. Use Eq. (5).
   Ans. $I' = 62.5 \times \frac{3}{4} + 90 \times \frac{1}{4} = 69.375$.
2. Check Prob. 1 by Eq. (6). *Ans.* $I' = 76.25 - 6.875 = 69.375$.
3. Find the moment of inertia of a 15- by 8-in. rectangle with respect to a diagonal. Solve by Eq. (5), regarding the axis as rotated about the center.
   Ans. $I' = 996.5 \text{ in.}^4$
4. Check Prob. 3 by means of two triangles with the diagonal of the rectangle as the common base.
5. Find the moment of inertia of a 4- by 3-in. rectangle with respect to an axis through the lower left corner at an angle of 20° with the 4-in. edge.
   Ans. $I' = 50 - 10.724 - 23.141 = 16.135 \text{ in.}^4$
6. Find the direction of the axes through the lower left corner of a rectangle 6 in. wide and 4 in. high for which the product of inertia is zero (Fig. 269). *Ans.* 30°28′ and 120°28′ with the 6-in. edge.

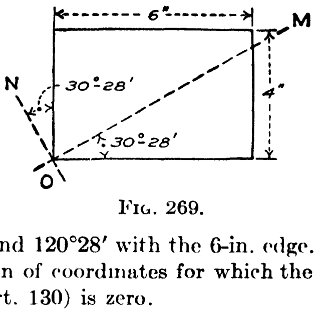

FIG. 269.

7. Find the direction of the axes through the origin of coordinates for which the product of inertia of the triangle of Prob. 5 (Art. 130) is zero.
   Ans. $\tan 2\theta = 3.2727$; $\theta = 36°30'$ and $126°30'$.

134. Relation of Moments of Inertia. If dA is an element of area in the XY plane, the moment of inertia of the entire area with respect to the X axis is

$$I_x = \int y^2\,dA \tag{1}$$

With respect to the Y axis,

$$I_y = \int x^2\,dA \tag{2}$$

The polar moment of inertia with respect to an axis through the origin is

$$J = \int r^2\,dA = \int (x^2 + y^2)\,dA = \int x^2\,dA + \int y^2\,dA = I_y + I_x \tag{3}$$

The sum of the moments of inertia of a plane area with respect to two axes in its plane which are perpendicular to each other is equal to the polar moment of inertia of the area with respect to the intersection of these axes. It follows that the sum of the moments of inertia of a plane area with respect to any pair of axes in its plane which are at right angles to each other and pass through the same point is constant.

Example

Find the moment of inertia of a 4- by 3-in. rectangle with respect to a line through one corner perpendicular to the diagonal.

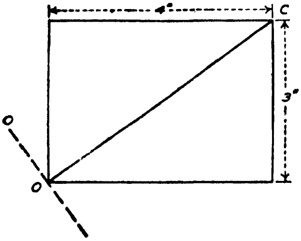

FIG. 270.

The diagonal *O-C* of Fig. 270 divides the rectangle into two triangles, each of which has a base of 5 in. and an altitude of 2.4 in. The moment of inertia of each triangle with respect to *O-C* is 5.76 in.^{4} and the moment of inertia of the rectangle is 11.52 in.^{4} If moments of inertia are taken with respect to horizontal and vertical axes through *O*,

$$I_x = 36 \qquad I_y = 64$$
$$I_x + I_y = J = 11.52 + I$$
$$I = 88.48 \text{ in.}^4$$

Problems

1. The moment of inertia of a circle with respect to a diameter is $\pi a^4/4$. By the principle of Eq. (3), show that the polar moment of inertia with respect to the center is $\pi a^4/2$.
2. The polar moment of inertia of a circle with respect to a point at its circumference is $3\pi a^4/2$. Find the moment of inertia of a circle with respect to a tangent in its plane. Check by the transfer of axis.
3. Find the moment of inertia of a 6- by 6- by 1-in. angle section with respect to the axis which bisects the angle between the legs of the section. Use Eq. (3) and get all the data from the handbook. Check by completing the square and subtracting the moment of inertia of the additional area from the moment of inertia of the square.

135. Mohr's Circle. The relations existing among the moments of inertia, the products of inertia, and the polar moment of inertia of a plane area with respect to axes through a point are so important and useful in practice that it is desirable to have some mnemonic device for keeping the formulas readily available for use without reference to notes or handbooks. Such a device is the graphical construction known (after its inventor) as Mohr's circle.[1]

If the axes *OX* and *OY* of Fig. 268 are the principal axes for the area, the product of inertia for these axes is zero and Eq. (6) of Art. 133 becomes

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[1] In addition to this problem in plane areas, two problems of strength of materials, one of mechanics, and one of statistics have solutions of the same mathematical form as this one and can be equally well referred to Mohr's circle. Mohr's circle for the formulas of "plane stresses" in strength of materials is in very general use.

$$I' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta$$

and Eq. (13) becomes

$$H' = \frac{I_x - I_y}{2} \sin 2\theta$$

Figure 271 shows these relations plotted on rectangular coordinates, with the ordinates representing values of H' and the abscissas representing values of I'. The angle 2θ locates a point on the circle corresponding to an axis revolved an angle θ from the axis of maximum moment of inertia of the area. The most usual practical problem is one in which I_x, I_y, H_x, and H_y are known for a pair of axes (not the principal axes) and the locations of the principal axes, $I_{\max}$, and $I_{\min}$, are required. The following example illustrates how such a problem is solved with the aid of Mohr's circle. The circle may be laid out to scale and the results determined graphically or the circle may be merely sketched and the desired values computed from the obvious relations.

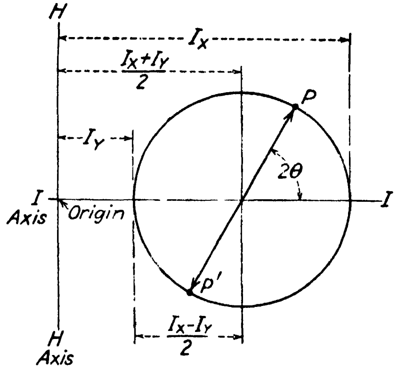

FIG. 271.

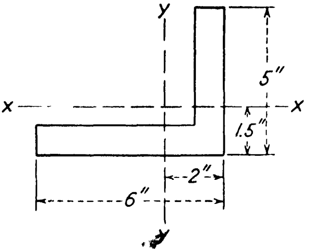

FIG. 272.

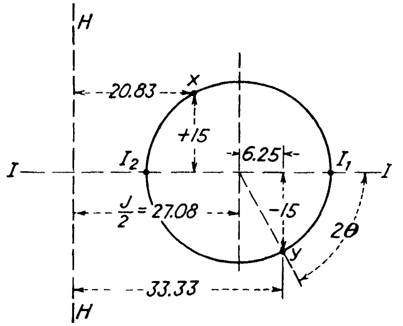

FIG. 273.

Example

Locate the principal axes and find the moments of inertia about these axes for the 6 in. by 5 in. by 1 in. angle section of Fig. 272. The axes shown pass through the centroid of the figure. $I_x = 20.83$ in.^{4}, $I_y = 33.33$ in.^{4}, $H_x = +15$, $H_y = -15$.

In Fig. 273, the point marked y has as its coordinates I_y and H_y, and the point marked x has the coordinates I_x and H_x. Since the X and Y axes in Fig. 272 are 90° apart, these two points are 180° apart and xy is a diameter of the circle with center at 27.08. The radius of the circle is $\sqrt{6.25^2 + 15^2} = 16.25$.

$$I_{\max} = 27.08 + 16.25 = 43.33 \text{ in.}^4 \qquad I_{\min} = 27.08 - 16.25 = 10.83 \text{ in.}^4$$

Point I_1 corresponds to the axis of $I_{\max}$. The angle 2θ is the angle whose tangent is $\frac{15}{6.25}$, $2\theta = 67°23'$; $\theta = 33°41.5'$. From point y to point I_1 in the circle, 2θ is counterclockwise, hence the Y axis in Fig. 272 must be rotated 33°41.5′ counterclockwise to place it in the position of the axis of maximum moment of inertia. The X axis, rotated through the same angle, becomes the axis of minimum moment of inertia.

Problems

1. Find the maximum and minimum moment of inertia of a 6- by 4-in. rectangle for axes through the lower left corner (Fig. 269).

From Prob. 6 of Art. 133, $\tan 2\theta = 1.800$; $2\theta = 60°57'$ and $240°57'$.

$I_{\min} = 208 - 80 \cos 60°57' - 144 \sin 60°57' = 43.27 \text{ in.}^4$

$I_{\max} = 208 - 80 \cos 240°27' - 144 \sin 240°57' = 372.73 \text{ in.}^4$

$I_{\min}$ is the moment of inertia with respect to OM of Fig. 269.

2. Find the direction of the axis through the center of gravity of a 6- by 5- by 1-in. angle section for which moment of inertia is a minimum and calculate the minimum moment of inertia and radius of gyration. Take data from previous problems and use Mohr's circle.

Ans. $\tan 2\theta = -2.4$; $I_{\min} = 27.083 - 2.404 - 13.846 = 10.833$.

3. For purlins and other special applications, rolled sections called "Z's" are sometimes used. The section resembles the letter Z except that the angles are right angles. The thickness of the upper and lower rectangles and of the longer rectangle connecting them is usually all the same. The standard aluminum 6-in. 4.61-lb (per lineal foot) Z has a total depth of 6 in. and upper and lower rectangles 3½ inches wide and ⅜ in. thick. Find I_x, I_y, and H for axes through the center of gravity parallel to the edges of the section. Lay out a Mohr's circle and determine the location of the principal axes and find the maximum and minimum moments of inertia. The Alcoa Structural Handbook gives $I_{\min}$ as 3.08 in.^{4} about an axis inclined at 26°55′ with the center line of the web.

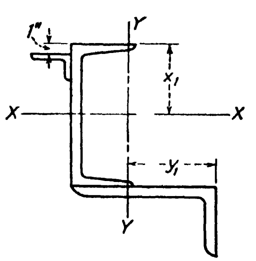

FIG. 274.

4. For a 6- by 4-in. by ½-in. angle section with axes parallel to the legs and through the center of gravity of the section, a handbook gives $I_x = 17.4$, $I_y = 6.3$, and the least radius of gyration as 0.87 in., without locating the axis of least moment of inertia. Use the given data to construct Mohr's circle and locate the principal axes. Find H.

Ans. Axis of $I_{\min}$ makes an angle of 23°45′ with the y axis.

5. The built-up section shown in Fig. 274 is called an *eave strut*. The 1-in. dimension is standard. Axes are through the center of gravity of the whole section. With channel 8 in., 11.5 lb, upper angle 3 by 3 by ¼ in., and bottom angle 4 by 3 by ¼ in., show that $I_x = 75.9 \text{ in.}^4$ and $I_y = 15.8 \text{ in.}^4$

6. Compute H for the section of Prob. 5, locate the axis of least moment of inertia, and find the minimum radius of gyration.

7. Find the maximum and minimum moments of inertia for axes through the centroid of an area composed of a vertical rectangle 2 in. wide and 12 in. deep to which are attached two semicircles of 3-in. radius, one with a diameter coincident with the right side of the rectangle and tangent to the top and the other coincident with the left side and tangent to the bottom.

CHAPTER 14

KINEMATICS AND DYNAMICS

136. Kinematics. Kinematics is the science of motion. The principal elements of kinematics are displacement (both linear and angular) and time. Dynamics, or kinetics, is the study of the relations among the forces acting on real masses and the changes in the state of motion of the masses. It is therefore necessary to understand something of kinematics in order to make a beginning in dynamics. In this chapter the simpler propositions of kinematics are introduced. These propositions are adequate for an understanding of dynamics and enable the student to arrive at the solutions of a large number of important practical problems. The general study of the kinematics of machinery is beyond the scope of this text.

In statics, every body considered was either at rest or in a state of uniform motion. The basic equations available for analysis were ΣF (in any direction) = 0 and ΣM (about any axis) = 0. The basic equations of dynamics are the same, except that the 0 of the right-hand member of the resolution equation is replaced by the mass of the body under consideration multiplied by the acceleration of its center of gravity in the direction of the resolution, and the 0 of the moment equation is replaced by the product of the moment of inertia of the body and its angular acceleration. Also, in dynamics, moments may not be taken about *any* axis but only about an axis through the center of gravity of the body and (sometimes) about an axis through the center of rotation. The laws of vector addition, subtraction, and multiplication are, of course, the same. The procedures of replacing a force by its components and of replacing a system of forces by its resultant which are found valid in statics are also valid in the dynamics of rigid bodies. Problems may be classified, as in statics, by concurrent and coplanar and their opposites, and the number of unknowns which can be determined in any class is the same as for the corresponding class of statics problem. The new element of *acceleration* may be either a known or an unknown. Energy equations are quite similar to the energy equations of statics but have the additional element of *kinetic energy*.

137. Displacement. When a body changes its position with respect to another body, it is said to be displaced relative to the other body. The distance from the initial position to the final position is the *displacement* of the body. Displacement is usually measured with respect to the earth but may be measured with respect to other bodies. Displacement has direction as well as magnitude and is, therefore, a vector quantity.

Problems

1. A man steps from a car and moves east a distance of 60 ft while the car moves east 150 ft. What is the displacement of the man relative to the earth and relative to the car? *Ans.* 60 ft east; 90 ft west.
2. A body moves from one position to another position which is 80 ft north and 150 ft east of the starting point. What is the displacement? *Ans.* 170 ft north; 61°56′ east.
3. A ball is struck by a bat Two seconds later it is 80 ft above, 48 ft north, and 36 ft east of the starting point. Find the displacement. *Ans.* 100 ft at an angle of 36°52′ with the vertical in a vertical plane which is north 36°52′ east.
4. A man starts from a point *A* and moves east 60 ft. At the same time a car starts from a point 20 ft north of *A* and moves north 60 ft. Find the displacement of the man with respect to the car. What is the change of displacement?

138. Velocity. Rate of displacement is called *velocity.* In problems of mechanics, velocity is generally expressed in feet per second. Since displacement is a vector, and velocity is displacement divided by time (which is not a vector) it follows that velocity is a vector. A velocity of 10 feet per second north is not equivalent to a velocity of 10 feet per second east. When only the magnitude of the motion is considered without reference to its direction, the rate of displacement is called *speed.* A body moving east at the rate of 10 feet per second has the same speed as a body moving north at the rate of 10 feet per second.

Velocity is calculated by dividing the difference in position by the corresponding difference in time. If x_1 is a coordinate of the position at the time t_1 and x_2 is a coordinate of the position at the time t_2, and if v_x is the component of the average velocity parallel to the X axis,

$$v_x = \frac{x_2 - x_1}{t_2 - t_1} \qquad \text{Formula XXV}$$

Similar expressions give the components of the velocity parallel to the other axes of coordinates. The actual average velocity is the vector sum of these components.

Formula XXV and the similar expressions give the *average velocity* during the time interval under consideration. In Fig. 275, a body may start at A and move along the circular path $ABCDE$, passing over each interval in 2 seconds. If the first 2-second interval is considered, the velocity during this interval is one-half of the length AB and is in the direction from A toward B. If the entire 8 seconds are considered, the apparent velocity is one-eighth of the length AE in the direction from A toward E. Also, the magnitude of the velocity may change. When a velocity varies in direction, in magnitude, or in both magnitude and direction, a very short interval of space and a correspondingly short interval of time are required to get a close approximation to the actual velocity at a particular time or place. Formula XXV then becomes

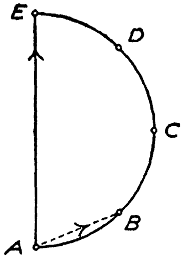

FIG. 275.

$$v_x = \frac{\Delta x}{\Delta t} \tag{1}$$

When the position of a body is given algebraically in terms of the time, Formula XXV may be written

$$v_x = \frac{dx}{dt} \tag{2}$$

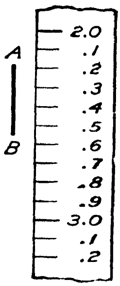

FIG. 276.

To determine experimentally the actual velocity at any given instant, means must be devised for measuring small intervals of time and corresponding increments of displacement. Suppose it is desired to find the velocity of a steel ball attached to a small parachute after falling 2 seconds from rest. The ball may fall beside a vertical scale as in Fig. 276. The time and displacement may be measured by means of a photographic camera provided with an "instantaneous shutter" of known time. An electromagnetic arrangement may be provided which will release the shutter 2 seconds after the ball starts to fall. If the ball is properly illuminated, its displacement is given by a vertical streak, AB of Fig. 213. Suppose that the "time" of the shutter is 0.02 second, and that the length of the streak on the plate is equivalent to 0.36 foot of the scale. The average velocity for the 0.02 second is

$$v = \frac{0.36}{0.02} = 18 \text{ ft per sec}$$

This experiment gives the average velocity for the interval of 0.02 second. If the change of velocity is all in one direction, *i.e.*, if the velocity is either increasing or decreasing throughout the entire interval, the average velocity is the velocity at about the middle of the interval. The result of this experiment is, then, the velocity at about 2.01 seconds after the beginning of the fall. If the velocity is desired at 2 seconds from the beginning of the fall, the apparatus must be arranged to release the shutter 1.99 seconds after the body starts to move.

Problems

1. The position of a body when a watch reads 7 sec is 32 ft from a given reference line. When the watch reads 9 sec it is 68 ft from the line, and when the watch reads 11 sec it is 104 ft from the line. What is the average velocity for each 2-sec interval?
2. The position of a body is given by $x = 2t^2$, in which t is the time in seconds and x is the position in feet with reference to some fixed line. What is the velocity when $t = 5$ sec? What is the velocity when $x = 72$ ft?
   Ans. 20 ft per sec; 24 ft per sec.
3. The position of a body is given by the expression $x = 2t^3$. An experimenter who does not know the equation of the motion of the body attempts to find approximately the velocity at $t = 4$ sec by measuring the distance the body travels from 3 sec to 5 sec. What result does he get if he measures correctly?
   Ans. $v_x = \frac{250 - 54}{5 - 3} = 98$ ft per sec.
4. If the equation were known, what result would be calculated for Prob. 3 by means of the derivative? *Ans.* $v_x = 96$ ft per sec.
5. Solve Prob. 3 by means of an interval of 0.2 sec. *Ans.* $v_x = 96.02$ ft per sec.

139. Acceleration. The rate of change of velocity is called the *acceleration*. If a body is moving east with a velocity of 10 feet per second, and 2 seconds later is moving east with a velocity of 40 feet per second, the change in velocity is 30 feet per second east. This change takes place in 2 seconds. The average rate of change is 30 ÷ 2 = 15 feet per second per second.

The expression for the component of the average acceleration in the direction of the X axis is given by

$$a_x = \frac{v_2 - v_1}{t_2 - t_1} \qquad \text{Formula XXVI}$$

in which v_1 is the component of the velocity in the direction of the X axis at the time t_1, v_2 is the component at the time t_2, and a_x is the acceleration.[1]

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[1] In some works of mathematical physics the components of velocity and acceleration are written thus: The component of the velocity parallel to the X axis is $\dot{x}$ and the component of the acceleration is $\ddot{x}$. A derivative with respect to time is written with a single dot over the variable. The second derivative with respect to time is written with two dots over the variable.

When the increment of time and the corresponding increment of velocity are very small, Formula XXVI becomes

$$a_x = \frac{dv_x}{dt} \tag{1}$$

Since $v_x = dx/dt$, Eq. (1) reduces to

$$a_x = \frac{d^2x}{dt^2} \tag{2}$$

The acceleration along the other coordinate axes is given by similar derivatives.

$$\begin{aligned} a_x &= \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \\ a_y &= \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \\ a_z &= \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \end{aligned} \qquad \text{Formulas XXVII}$$

Problems

1. The position of a body is given by $x = kt^2$. When the displacement varies as the square of the time, show that the velocity varies as the time and that the acceleration is constant.
2. The position of a body is given by the equation $x = 3t^2 + 2t + 6$. Find the velocity and acceleration when $x = 91$ and when $x = 167$.
   Ans. 32 ft per sec; 44 ft per sec; 6 ft per sec per sec.
3. The position of a body is 59 ft at 5 sec, 77 ft at 7 sec, 105 ft at 9 sec, and 143 ft at 11 sec. Assuming that the average velocity is the velocity at the middle of an interval, find the velocity at 6, 8, and 10 sec and find the acceleration.
4. The position of a body is 45 ft at 3 sec, 81 ft at 5 sec, and 297 ft at 9 sec. Find the acceleration. *Ans.* 12 ft per sec per sec.
5. A car running down an inclined track went 20 ft in the first second and 70 ft in the next 2 sec. Assume that the acceleration was constant and find the initial velocity and the acceleration.
   Ans. Initial velocity = 15 ft per sec down the plane; acceleration = 10 ft per sec per sec.

140. Constant Acceleration. Many practical problems of mechanics deal with bodies which move for finite intervals of time under the action of systems of forces which do not change during the time. Such bodies have constant accelerations, and problems involving their positions and velocities can be readily solved by algebraic equations resulting from integrating the equations of Formulas XXVII.

$$\frac{dv}{dt} = a \tag{1}$$

Multiplying by dt and integrating,

$$v = at + C_1 \tag{2}$$

in which C_1 is an integration constant. If v_0 is the velocity when $t = 0$, $C_1 = v_0$ and Eq. (1) becomes

$$v = v_0 + at \tag{3}$$

Substituting $v = ds/dt$ in Eq. (3),

$$\frac{ds}{dt} = v_0 + at \tag{4}$$

$$s = v_0t + \frac{at^2}{2} + C_2 \tag{5}$$

If space is measured from some point $s = s_0$ when $t = 0$, then $C_2 = s_0$, and Eq. (5) reads

$$s = s_0 + v_0t + \frac{at^2}{2} \tag{6}$$

If s is measured from the position where $t = 0$, then $C_2 = 0$, and

$$s = v_0t + \frac{at^2}{2} \tag{7}$$

If s is measured from the position where $t = 0$ and if the velocity is zero at this position, then $C_2 = 0$, $v_0 = C_1 = 0$, and Eq. (6) becomes

$$s = \frac{at^2}{2} \tag{8}$$

Displacement, velocity, and acceleration are all vector quantities. Equations (2) through (8) can be used as algebraic equations only when the initial velocity, the initial displacement, and the acceleration are all in the same direction. The opposite direction can be taken care of by negative signs in the same equations, but some other direction, such as a direction perpendicular to the chosen one, must be handled (if these equations are to be used) by breaking the initial velocity and the acceleration into components and treating the two sets of components separately as though there were two independent motions. The final total results are obtained by vector addition of the separate components.

Example

A body is moving east 20 ft per sec. It is moving north 20 ft per sec 2 sec later. Find the average acceleration during the 2-sec interval.

Figure 277 is the vector diagram of velocities. The increment of velocity is Δv, which must be added to v_1 to get v_2. The vector equation is

$$v_1 + \Delta v = v_2 \qquad v_2 - v_1 = \Delta v$$

By trigonometry,

$\Delta v = 28.28$ ft per sec northwest.

$a = \frac{28.28}{2} = 14.14$ ft per sec per sec directed northwest.

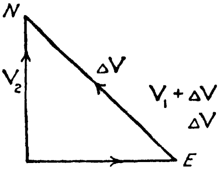

FIG. 277.

Problems

1. The velocity of a body is changed from 50 ft per sec east to 120 ft per sec north in 5 sec. Find the direction and magnitude of the acceleration. Draw the vector diagram of velocity and change of velocity.
Ans. North 22°37′ west; 26 ft per sec per sec.
2. A ball is thrown against a wall with a velocity of 100 ft per sec and rebounds in the opposite direction with a velocity of 90 ft per sec. If the time of contact is 0.001 sec, what is the acceleration? *Ans.* $a = 190{,}000$ ft per sec per sec.
3. A smooth rectangular plane 12 ft by 18 ft is placed with the 18-ft edges horizontal and the 12-ft edges inclined so that a body anywhere on the plane is accelerated down the plane at 16 ft per sec per sec. A body is projected from one of the lower corners toward the mid-point of the upper edge, with an initial velocity of 30 ft per sec. Find where and when it will leave the plane.
Ans. Off the upper edge 2.412 ft from the mid-point in 0.634 sec.
4. Solve Prob. 3 if the initial velocity is 20 ft per sec.

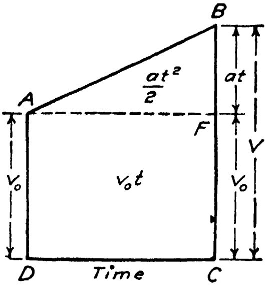

FIG. 278.

Figure 278 shows Eq. (3) plotted with time as abscissas and velocity as ordinates. The velocity curve is the straight line *AB* with slope equal to the acceleration. Equation (7) is represented by the area under the line *AB*. When the acceleration is constant, the average velocity for any time interval is the actual velocity at the middle of the time interval, or one-half the sum of the ordinates *DA* and *CB*. From a study of the figure, the following rules can be seen to be true:

1. $\text{Average velocity} = \frac{\text{initial velocity} + \text{final velocity}}{2}$.
2. Average velocity occurs at mid-time.

3. Average velocity × time = displacement (for the time interval).

4. Acceleration = $\dfrac{\text{final velocity} - \text{initial velocity}}{\text{time}}$.

These rules can be used instead of Eqs. (1), (3), (7), and (8). They usually lead to quicker solutions than do the use of the equations, and the student is likely to get a better sense of the physical ideas from their use. With variable accelerations, the line AB of Fig. 278 is curved, *and the method of calculus must be used except in the case of simple curves whose properties are familiar.*

Problems

(Solve Probs. 5 to 11 without writing.)

5. The velocity of a body changes from 30 ft per sec east to 50 ft per sec east in 5 sec. What is the average velocity? How far does the body travel in the 5 sec? *Ans.* 40 ft per sec; 40 × 5 = 200 ft.
6. What is the acceleration of the body of Prob. 5? What is the average velocity for the first 3 sec? How far does it travel the first 3 sec? What is the average velocity the last 2 sec? How far does it travel the last 2 sec?
7. A body moving with a velocity of 60 ft per sec is brought to rest in 4 sec by a constant force. How far does it go? *Ans.* $s = 30 \times 4 = 120$ ft.
8. A body moving with a velocity of 60 ft per sec is brought to rest in 180 ft by a constant force. What is the acceleration?

The average velocity is 30 ft per sec. The time is $^{180}/_{30} = 6$ sec. The change in velocity is -60 ft per sec in 6 sec.

Ans. $a = -^{60}/_{6} = -10$ ft per sec per sec.

9. A body has its velocity changed from 50 ft per sec east to 20 ft per sec east while moving 210 ft. Find the acceleration.
10. A body has its velocity changed from 60 ft per sec east to 40 ft per sec west in 5 sec. How far is it from the starting point at the end of the 5 sec? How far does it actually travel in the 5 sec? How far does it travel the last 3 sec of the time? Which one of these answers is given correctly by Eq. (3)? Explain.
11. A mass of 40 lb is moving east with a velocity of 60 ft per sec. After an interval of 4 sec, it is 160 ft east of the starting point. Find the final velocity and the acceleration. Solve by means of average velocity and the definition of acceleration. Check by Eqs. (3) and (7).
12. A body having a positive acceleration of 20 ft per sec per sec passes a point A, at which $s = 60$, with a positive velocity of 30 ft per sec. If time is measured from point A, write the equation of motion *Ans.* $s = 60 + 30t + 10t^2$.
13. Write the equation of motion of Prob. 12, beginning to count time 1 sec after the body has passed the point A. *Ans.* $s = 100 + 50t + 10t^2$.
14. Solve Prob. 12 if the velocity at A is -30 ft per sec.

141. Curvilinear Motion. The equations for motion with constant acceleration along a straight line may be applied equally well to motion along a curve provided the acceleration is at all times in the direction (either positive or negative) of the velocity, *i.e.*, tangential to the curve. This component of acceleration is called the *tangential acceleration,* and there will invariably be associated with it an acceleration toward the center of curvature called the *normal acceleration.* The normal acceleration of a point moving along a curve arises from the change of direction of the velocity rather than from a change of speed. Regardless of whether the speed is changing positively, negatively, or not at all, *the normal acceleration is always the square of the speed divided by the radius of curvature.* This proposition is of the utmost importance in mechanics. It can be proved in a number of ways. The first proof below is selected for its simplicity, and the second for its generality. For practical purposes, the student may take his choice, but in any case he should strive for a thorough understanding.

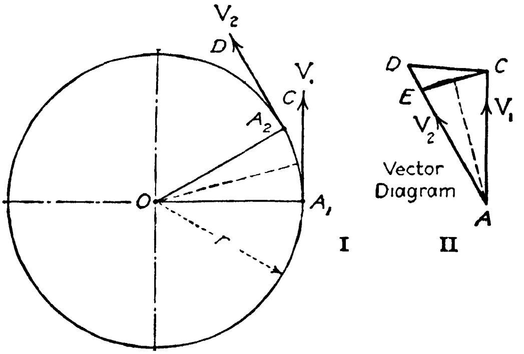

FIG. 279.

Figure 279 represents a particle which is moving, with increasing speed, along the circumference of a circle of radius r. At the point A_1 its linear velocity is v_1. At A_2, its linear velocity is v_2. The direction and magnitude of the velocity at A_1 are shown by the vector A_1C, and the direction of the velocity at A_2 is shown by the vector A_2D. Figure 279,II is the vector diagram of the velocities with a common initial point A. The vector CD of Fig. 279,II represents the change in velocity when the body moves from A_1 to A_2. This increment of velocity may be resolved into two components CE and ED. The length AE is made equal to AC. The component ED represents the increment which is due to change in the magnitude of the velocity; the component CE represents the increment which is due to change in the direction. Since AEC is an isosceles triangle, the direction of CE is normal to the line which bisects the angle CAE and is parallel to the line which bisects the angle A_1OA_2 of Fig. 279,I.

If v is the magnitude of the velocity at A_1 and if the distance from A_1 *to* A_2 *is infinitesimal, the increment CE* is

$$dv = v\,d\theta \tag{1}$$

in which $d\theta$ is the change in the direction of the velocity and dv is the increment which is due to the change in direction. Since the direction of this increment is perpendicular to the line which makes an angle $d\theta/2$ with the direction of v_1, this change of velocity is toward the center of the circle of Fig. 279,I.

The linear acceleration toward the center is

$$a = \frac{dv}{dt} = \frac{v\,d\theta}{dt} \tag{2}$$

Since $A_1A_2 = r\,d\theta = v\,dt$,

$$\frac{d\theta}{dt} = \frac{v}{r}$$

$$a = \frac{v^2}{r} \qquad \text{Formula XXVIII}$$

As will be shown in Art. 142, $v = r\omega$,

$$a = r\omega^2 \tag{3}$$

For the special case of constant speed, ED would be zero, and for decreasing speed ED would be negative, *i.e.*, D would be between E and A. In any case, the derivation of the formula for the normal acceleration is unchanged.

Problems

1. A particle is whirled in a circular path 8 ft in diameter with a speed of 20 ft per sec. What is its acceleration toward the center?

Ans. $a = 100$ ft per sec per sec.

2. Solve Prob 1 if the diameter is 4 ft instead of 8 ft.

3. A wheel 4 ft in diameter is making 300 rpm. What is the acceleration of a point on the rim toward the center?

Ans. $a = 2 \times 10^2 \times \pi^2 = 1974$ ft per sec per sec.

When a point is moving in a circle, the acceleration toward the center of the circle is at right angles to the direction of motion and this is a special case of normal acceleration. *When a particle moves along any curve, the normal acceleration at any point is* v^2/ρ, *in which* ρ *is the radius of curvature of the curve at that point. The tangential acceleration is the rate of change of speed.*

In Fig. 280, *ABD* is a curve and *EB* is a line tangent at *B*. In Fig. 280,I, ϕ is the angle which the tangent makes with the *X* axis *OX*. At *B*,

$$\frac{dy}{dt} = \frac{ds}{dt} \sin \phi \tag{4}$$

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \sin \phi + \frac{ds}{dt} \cos \phi \frac{d\phi}{dt} \tag{5}$$

Since

$$\frac{d\phi}{dt} = \frac{d\phi}{ds} \times \frac{ds}{dt}$$

Eq. (5) may be written

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \sin \phi + \left(\frac{ds}{dt}\right)^2 \cos \phi \frac{d\phi}{ds} \tag{6}$$

The coordinate axes may be taken in any direction. In Fig. 280,II the *X* axis is normal to the tangent *EB*. The angle ϕ is $\pi/2$ and Eq. 6 becomes

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \tag{7}$$

Since the *Y* axis is now taken parallel to the tangent, Eq. (7) gives the acceleration along the curve. *The tangential acceleration is the rate of change of speed.* The line *OY′* of Fig. 280,II may be taken as the *X* axis, and a line opposite the direction of *OX′* will now be the *Y* axis. Since the *X* axis is now parallel to the tangent *EB*, the angle ϕ is zero. Equation (6) now is

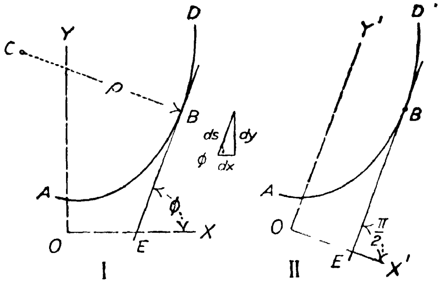

FIG. 280.

$$\frac{d^2y}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d\phi}{ds} = v^2 \frac{d\phi}{ds} \tag{8}$$

If ρ is the radius of curvature, $\rho\, d\phi = ds$ and $d\phi/ds = 1/\rho$. When this is substituted in Eq. (8),

$$\frac{d^2y}{dt^2} = \frac{v^2}{\rho}$$

The normal acceleration at any point is the square of the speed at that point divided by the radius of curvature. From the calculus,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

142. Angular Displacement and Velocity.

When a rigid body rotates about an axis, all parts of the body turn through the same angle. In Fig. 281, the axis passes through O and is perpendicular to the plane of the paper. If a point on the body moves from position B to position B', its path is the arc BB'. The angular displacement of this point is the angle between the radii OB and OB'. If r is the length of the radius from O to B, the angular displacement in radians is

$$\theta = \frac{\text{arc } BB'}{r}$$

Angular displacement is linear displacement divided by the radius.

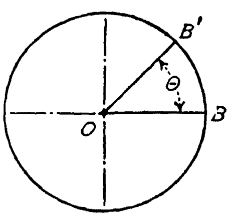

FIG. 281.

Angular velocity is the rate of change of angular displacement. Angular velocity is usually represented by the Greek letter ω (omega).

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1} \qquad (1)$$

Equation (1) is analogous to the corresponding formula for linear velocity. For small intervals of time,

$$\omega = \frac{d\theta}{dt} \qquad (2)$$

If v is the linear velocity of a point on a rotating body, vt is the length of the arc which the point describes in time t. The angular displacement of the point is vt/r. The angular velocity is the angular displacement in radians divided by the time

$$\frac{vt/r}{t} = \frac{v}{r} = \omega \qquad (3)$$

$$v = r\omega \qquad \text{Formula XXIX}$$

Problems

1. A point moves a distance of 6 ft in the circumference of a circle of 3-ft. radius. Find the angular displacement in radians. *Ans.* $\theta = 2$ rad.
2. A body revolves once on its axis. What is the angular displacement? *Ans.* $\theta = 2\pi = 6.2832$ rad.

3. What is the angular displacement when a body turns through an angle of 60°?
4. A cylinder 4 ft in diameter turns through an angle of 50°. Find the linear displacement of any point on the surface of this cylinder.
5. The hour hand of a clock moves from 2 to 10. What is the angular displacement?
6. The circumference of a wheel 4 feet in diameter is moving 1800 ft per min. What is the angular velocity of the wheel? *Ans.* 15 rad per sec.
7. What is the angular velocity of a pulley which is making 300 rpm? *Ans.* $10\pi = 31.416$ rad per sec.
8. What is the angular velocity of an automobile which is running on a circular track of a 200-ft radius with a speed of 30 mph?

143. Angular Acceleration. The rate of change of angular velocity is the angular acceleration.

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

in which α is the angular acceleration. Angular acceleration is expressed in radians per second per second. For small intervals of time, the angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The expressions for angular velocity and displacement when the angular acceleration is constant are similar to those for linear velocity and displacement when the linear acceleration is constant [Eqs. (3) and (6), Art. 140].

$$\omega = \omega_0 + \alpha t \tag{1}$$

$$\theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2} \tag{2}$$

in which ω_0 is the initial velocity and θ_0 is the displacement from the reference position at the initial time.

The first equation above may be derived from the definition of acceleration. The second equation may be derived by multiplying the average velocity by the time and adding θ_0.

Problems

1. A wheel has an acceleration of 3 rev per sec per sec. Its initial velocity is 5 rev per sec. What will be its velocity after 6 sec? *Ans.* $\omega = 23$ rev per sec.
2. How many revolutions will the wheel of Prob. 1 make during the 6-sec interval? How many during the last 4 sec of the 6? *Ans.* 84; 68.
3. Write Eq. (2) for the last 4 sec of Prob. 2, assuming that time began at the end of the second second.

4. A wheel is making 100 rpm. After an interval of 20 sec it is making 1900 rpm. Find the angular acceleration in radians per second per second.

Ans. $\alpha = 3\pi = 9.425$ rad per sec per sec.

144. Relations of Linear and Angular Motion.

Machines, even simple machines, normally consist of sets of connected bodies, some of which have linear motions while others have angular motions. Many machines have also elements which appear to have both linear and angular motions at the same time. Such motions can normally be reduced at any instant to angular motions or rotations about points called *instantaneous centers*. The more complicated of these motions are properly left for the special study of kinematics of machines, but a knowledge of at least the simpler relations is necessary for an understanding of dynamics. With a few exceptions, the relations given below are the only ones used in this text.

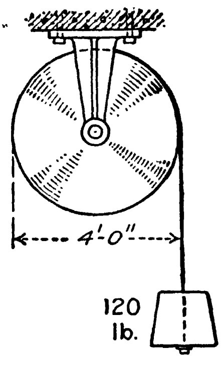

FIG. 282.

Figure 282 represents a drum mounted on a central shaft. A cord attached to the drum and wound around it has a mass of 120 pounds hanging from the free end. The radius of this drum is 2 feet, and if the drum is turned clockwise through an angle of θ radians, the suspended mass will descend 2θ feet. This relation is expressed by the equation

$$s = r\theta \tag{1}$$

in which s is the displacement, r the radius of the drum, and θ the angular displacement in radians. If r is in feet, s has the same units. Differentiating Eq. (1) gives the velocity relations in this case,

$$v = r\omega \tag{2}$$

and a second differentiation gives the relation of accelerations

$$a = r\alpha \tag{3}$$

In addition to expressing the relations between the linear motion of the mass of Fig. 282 and the angular motion of the drum, Eqs. (1), (2), and (3) serve to describe the motion of any point on or in the drum in terms of the angular motion of the drum. If r is taken as the distance from the center to any point, the displacement (along its circle) of any point of the drum is $r\theta$, the tangential velocity of the point is $r\omega$, and the tangential component of the acceleration of the point is $r\alpha$.

Figure 283 represents a cylindrical body with center at O rolling along a horizontal surface. In this case, the point of contact B is an instantaneous center. The point O moves in a straight line parallel to the surface on which the cylinder rolls. If ω is the angular velocity of the cylinder and α its angular acceleration, the linear velocity of O is $r\omega$, r being the radius BO, and the acceleration of this point is $r\alpha$. Relative to an observer stationed at O, the plane is coming off tangentially from the drum, just as was the rope of Fig. 282. The arrow at O shows the direction of motion of this point. The other three arrows show the directions of motion of other points relative to an observer at rest with respect to the horizontal surface. Each of the points has a velocity of $r\omega$ when the angular velocity of the cylinder is ω and r is the distance from B to the point in question.

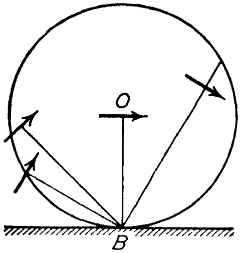

FIG. 283.

If the path along which the cylinder rolls is the arc of a circle, then the center O moves in a circular arc with a radius greater or less by a distance BO than the radius of the arc on which the cylinder rolls. Equations (1), (2), and (3) still describe the motion of the point O along its path. In this case O can have no displacement or velocity normal to its path, but there is an *acceleration* normal to its path equal to its speed squared divided by the radius of curvature of the path, as stated in Formula XXVIII. Equation (3) of Art. 141 is Formula XXVIII in another form. In applying Eq. (3) of Art. 141 to find the normal acceleration of the center of a body rolling along a curve, the student should use extreme care, since the r and the ω of that equation are not the radius and the angular velocity of the rolling body.

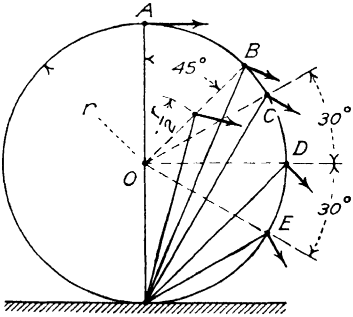

FIG. 284.

Problems

1. An elevator cage descends at 40 ft per sec. What is the angular velocity of the cable drum if its diameter is 3 ft?
2. Find the angular acceleration of the drum of Prob. 1 if the elevator is brought to a stop uniformly in 4 sec.
3. What is the angular velocity of an automobile wheel 26 in. in diameter when the car is running at 60 mph?
4. In Fig. 284, find the direction and magnitude of the velocity of each of the points A, B, C, D, and E. Solve by regarding the axis as moving in a straight line while the points revolve about it.

5. Solve Prob. 4 by considering the cylinder as rotating about the line of contact with the plane.

Rotation of a body about any axis with an angular velocity of ω radians per second is equivalent to a linear velocity of $r\omega$ feet per second combined with an angular velocity of ω radians per second about a parallel axis through the center of gravity. Figure 285 shows four positions of a body AB which rotates about an axis O normal to the plane of the paper. The axis through the center of gravity of AB passes through C. At position 1 at the right of the figure, the point B is to the right of C. At the second position, B is above C. At the third position, B is to the left of C. At the last position, B is below C. It is evident, therefore, that when the body makes one revolution about O, it makes one revolution about C.

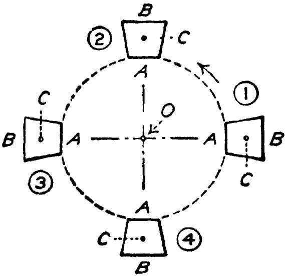

FIG. 285.

Problems

6. Find the angular velocity of an automobile which is proceeding at 60 mph around a curve of 440-ft radius. *Ans.* 0.20 rad per sec.
7. Find the normal acceleration of the car of Prob. 6.

145. Acceleration of a Mass Center. Many equations of dynamics are closely analogous to equations of resolution in statics. All the forces acting on a free body are resolved in some chosen direction just as in statics, but instead of equating the sum to zero it is equated to the mass of the free body multiplied by the component in the chosen direction of the acceleration of the mass center of the free body. This procedure is correct regardless of the complications of the motion of the free body and regardless of whether the free body is a single rigid body, a nonrigid body, or even a collection or group of bodies with or without connections. The correctness of this procedure is proved by Newton's second law of motion in connection with the following important proposition: *The product obtained when the mass of a body is multiplied by the acceleration of its center of gravity is equal to the sum of the products obtained when the mass of each element of the body is multiplied by its acceleration.*

If m is the mass of a body and x is the distance of its center of gravity from the YZ plane, and if m_1, m_2, m_3 . . . , represent the masses of elements or particles of the body, and x_1, x_2, x_3, respectively, represent the distances of these elements from the *YZ* plane, the moment equation gives

$$m\bar{x} = m_1x_1 + m_2x_2 + m_3x_3 + \cdots \tag{1}$$

$$m\frac{d\bar{x}}{dt} = m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} + m_3\frac{dx_3}{dt} + \cdots \tag{2}$$

$$m\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + m_3\frac{d^2x_3}{dt^2} + \cdots \tag{3}$$

Equation (3) proves the proposition. Similar equations may be written for the accelerations along the *Y* axis and along the *Z* axis.

The masses m_1, m_2, . . . , may represent elements of a rigid body or separate particles. The equation applies to a system of relatively large bodies, provided x_1, x_2, . . . , each represents the coordinate of the center of gravity of one of the bodies.

CHAPTER 15

FORCE AND MOTION

146. Newton's Laws of Motion. Newton's laws of motion state the relation of the forces which act on a body or a system of bodies to the changes of motion which these forces produce. Use is made of the idea of *momentum*, which is sometimes called the *quantity of motion*. If m is the mass of a body and v is its velocity,

$$\text{Momentum} = mv \qquad \text{Formula XXX}$$

The change of momentum of a given body is proportional to the change of velocity, since the mass of the body remains constant. Since rate of change of velocity is acceleration, rate of change of momentum is proportional to the mass multiplied by the acceleration.

Equation (2) of Art. 145 shows that the product of the mass of a body multiplied by the velocity of its center of gravity is equal to the sum of the momenta of the elements or particles which make up the body. The equation applies to the elements of a single rigid body, or to a group of separate bodies which form a system.

Newton's laws are

1. *Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.*
2. *Rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.*
3. *To every action there is always an equal and opposite reaction.*

The first law is really a special case of the second law, change of momentum is proportional to the impressed force. In the special case in which the impressed force is zero, the change of momentum is zero.

The third law applies to the mutual action of two bodies at their *surface of contact.* A book resting on a table pushes down on the table; the table pushes up on the book an equal amount. Action and reaction must not be confused with equilibrium of a free body. The book as a free body is in equilibrium under the reaction of the table and the attraction of the earth. There is action and reaction between the table and the book, and there is another action and reaction between the book and the ether which is supposed to transmit the attraction of the earth. It is currently fashionable to be skeptical about the ether and to consider instead that there is, in the nature of things, an acceleration of the book toward the earth. The upward force of the table on the book gives the book an equal upward acceleration, thus leaving it with a net acceleration of zero. That this is not nonsense is due to the fact that force and acceleration are indistinguishable mathematically. It will probably be better for the student to adopt the older view that the earth attracts bodies with forces, called *weights*, which are in proportion to the masses of the bodies, and that these forces, when they alone act on the bodies, give the bodies near the surface of the earth an acceleration of about 32.2 feet per second per second downward.

147. Force and Acceleration. Newton's second law states the experimental fact that the rate of change of momentum varies as the resultant impressed force. The law applies to a single particle, or to a large body or system of bodies, and the momentum is the product of the mass multiplied by the velocity of the center of gravity.

Since the rate of change of momentum is proportional to the acceleration, the second law may be written in the form of an equation,

$$\text{Effective force} = \text{a constant} \times \text{the mass} \times \text{the acceleration}$$

$$\text{Force} = k \times \text{mass} \times \text{acceleration} \qquad \text{Formula XXXI}$$

in which k is a constant which depends upon the units of length, mass, force, and time which enter into the equation.

Units of force or mass have been devised which make the constant k equal to unity. With such units, Formula XXXI becomes

$$\text{Force} = \text{mass} \times \text{acceleration} \qquad \text{Formula XXXII}$$

148. Gravitational Units of Force. A gravitational unit of force is the weight of a unit mass. A *pound force* is the *weight* of a *pound mass.* A *kilogram force* is the weight of a *kilogram mass.* These are the units of force commonly employed in engineering work. The mass of a body in pounds is numerically equal to its weight in pounds.

When a body falls in a vacuum, the effective force is the *weight* of the body. The acceleration is approximately 32.2 feet per second per second, which is represented in equations by the letter g. If W represents the weight of a body in pounds and m represents its mass in pounds, substitution in Formula XXXI gives

$$W = kmg \tag{1}$$

Since W is numerically equal to m, the equation may be divided by W,

$$1 = kg \tag{2}$$

$$k = \frac{1}{g} \tag{3}$$

Substitution of this value of k in Formula XXXI gives

$$P = \frac{ma}{g} \qquad \text{Formula XXXIII}$$

in which P is the effective force in pounds, m is the mass in pounds, a is the acceleration in feet per second per second, and g is a number (not an acceleration).

Since the weight of a body in pounds is equal to its mass in pounds, m in Formula XXXIII may be replaced by W.

$$P = \frac{Wa}{g} \qquad \text{Formula XXXIV}$$

in which W is the *mass* in pounds, P is the effective force in pounds, a is the acceleration in feet per second per second, and g is a number.

Since m is frequently used for another unit of mass and since the equation is often printed in the form of Formula XXXIV, this form will be given the preference. It will be understood that W means mass in pounds (weight in the ordinary commercial sense), and Formula XXXIV is identical with Formula XXXIII.

The acceleration of gravity varies with the latitude and altitude. For Great Britain, the average is about 32.2 feet per second per second. The value of g is generally given as 32.2 in English books. At 40° latitude, which may be taken as the average for the United States, the acceleration of gravity is about 32.16. This figure is largely used in American publications. The acceleration at the sea level at 45° latitude has been suggested as the *standard*. This is 32.174 feet per second per second. The corresponding value in the metric system is 980.66 centimeters per second per second.

Using the standard value of g, Formulas XXXIII and XXXIV are written

$$\text{Force in pounds} = \frac{\text{mass in pounds} \times \text{acceleration in feet per second per second}}{32.174}$$

Formula XXXIV applies also to the metric system provided corresponding units of mass, weight, and acceleration are substituted.

$$\text{Force in grams} = \frac{\text{mass in grams} \times \text{acceleration in centimeters per second per second}}{980.66}$$

$$\text{Force in kilograms} = \frac{\text{mass in kilograms} \times \text{acceleration in meters per second per second}}{9.8066}$$

Problems

1. A mass of 60 lb, moving with a velocity of 40 ft per sec, is brought to rest by a constant retarding force in 5 sec. Find the acceleration and the effective force. *Ans.* $a = -8$ ft per sec per sec; $P = 14.92$ lb.
2. A mass of 40 lb is moving on a horizontal plane with a velocity of 60 ft per sec. The coefficient of friction is 0.3. Find the acceleration and the time required to stop the body. *Ans.* $a = -9.65$ ft per sec per sec; $t = 6.218$ sec.
3. A mass of 40 lb is placed on a smooth inclined plane which makes an angle of 25° with the horizontal. What is the effective force down the plane? What is the acceleration? *Ans.* $P = 16.905$ lb; $a = 13.59$ ft per sec per sec.
4. Solve Prob. 3 if the coefficient of friction is 0.3. *Ans.* $P = 6.030$ lb; $a = 4.85$ ft per sec per sec.
5. Solve Prob. 3 if the coefficient of friction is 0.45.
6. A mass of 75 lb has an initial velocity of 50 ft per sec east. It is subjected to a constant force of 18 lb directed north. Find its position and velocity at the end of 4 sec.

   Ans. Displacement = 209.32 ft north 72°50′ east; velocity = 58.77 ft per sec north 58°18′ east.
7. A mass of 20 lb moving on a horizontal plane with a velocity of 60 ft per sec is brought to rest in 150 ft by the friction of the plane. Find the coefficient of friction. *Ans.* $\mu = 0.373$.
8. Solve Prob. 7 if the body is brought to rest in 6 sec. How far does it travel in 6 sec?
9. An automobile with all wheels sliding skids 68 ft to rest in 3.2 sec on a level road. What was its initial speed? What is the coefficient of friction between the tires and the pavement? *Ans.* $\mu = 0.413$.
10. A body starts up a 20° plane with a velocity of 80 ft per sec. The coefficient of friction is 0.2. What is the negative acceleration when the body is going up the plane? In what time will it come to rest and what will be its distance from the initial point? What is the acceleration when the body is coming down the plane? How long will it take to come down and what will be the final velocity at the bottom?

    Ans. Distance up = 187.68 ft; maximum velocity down = 43.195 ft per sec.
11. Solve Prob. 10 if the coefficient of friction is 0.38.

149. Other Systems of Units. Formula XXXIV is an equation expressing Newton's second law with a set of units called a *gravitational system*. In this equation, P is the force required to accelerate a mass of W pounds at a rate a and g is equal to the acceleration of gravity at the same location. The gravitational force W varies from place to place on the earth since it is the force required to lift W *standard* pound masses. It is mostly a matter of academic interest, but the engineer's unit of work, the *foot-pound,* varies with the location on earth.

A system of units devised to be invariable with the position on earth (or even out on the moon) is called an *absolute system.* The unit of force in such a system is defined by the units of mass, length, and time. In the English absolute system, the *poundal* is defined as the force which will impart to the standard pound a velocity of 1 foot per second in 1 second. This system is rarely used by anybody, since modern physicists use the metric system, and the poundal, which is a force about sufficient to lift ½ ounce, is too small for most engineering work. In the metric system, the *dyne* is defined as the force required to impart to a mass of 1 gram a velocity of 1 centimeter per second in 1 second. This is a very small force indeed, but it is a useful unit in physical work. Engineers who use the metric system usually use the gravitational units.

The quantity W/g of Formula XXXIV is sometimes considered the mass in *slugs* or *geepounds.* With this concept of a unit of mass, the system is called the *British engineers' system.*

$$\text{Force} = \text{mass} \times \text{acceleration}$$

with force in pounds, mass in slugs, and acceleration in feet per second per second. There is now coming into popular usage a unit of acceleration called the *gee* which is about 32.2 feet per second per second. Then force = mass × acceleration, with force in pounds, mass in pounds, and acceleration in gees. The unit was proposed long ago but never became popular until recently. It appeals to the common sense, and every aviator knows that if he makes a turn with an acceleration of 5 gees he feels himself pressed five times as tightly to his seat as he is normally.

150. Resolution Equations of Dynamics. In the preceding article, the force P of Formulas XXXIII and XXXIV was referred to as the *effective force.* It is the *resultant* of the force system acting on the free body. The acceleration of the center of gravity of the body is in the direction of this resultant and the magnitude of the acceleration is in proportion to the magnitude of this resultant. It is not necessary to resolve in the direction of the acceleration. Any convenient direction may be chosen, but the a of the formula must be the *component of* *acceleration of the center of gravity of the body* in the chosen direction. The effective force P is the sum of the components in the chosen direction of all forces acting on the body. It is often convenient to resolve in the direction of an acceleration, known or assumed, and in a direction in which there is known to be no acceleration, as was done in Probs. 2, 3, 7, etc., of Art. 148. In this case the second equation is exactly like the ones used in statics, the right-hand member of Formula XXXIV being zero. The following examples and problems are designed to give the student some experience in the use of Formula XXIV. When the force system under consideration is a coplanar one, two independent resolution equations can always be written.

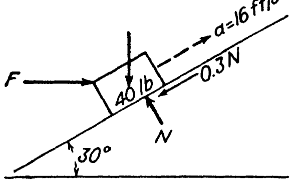

FIG. 286.

Example 1

Find the constant horizontal force required to start a 40-lb body up a 30° inclined plane and cause it to go 8 ft in the first second if the coefficient of friction between the body and the plane is 0.3.

Figure 286 shows the free-body diagram. From the displacement and time data the acceleration has been found to be 16 ft per sec per sec up the plane. This is indicated in the sketch; otherwise the diagram is exactly the same as if the problem were one of statics. A body moving in a straight line has zero acceleration in a direction perpendicular to that line.

Resolving in a direction perpendicular to the plane,

$$N - 40 \cos 30° - F \cos 60° = 0 \qquad (1)$$

Resolving parallel to the plane,

$$F \cos 30° - 40 \cos 60° - 0.3N = \frac{40 \times 16}{32.17} \qquad (2)$$

Solving simultaneously,

$$F = 70.2 \text{ lb}$$

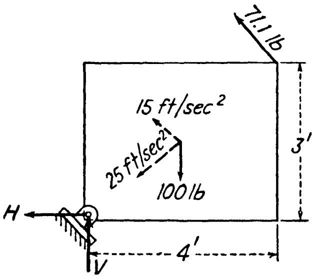

FIG. 287.

Example 2

Figure 287 shows a rectangular board 3 by 4 ft hinged at the lower left corner. The board weighs 100 lb and is being turned about the hinge by the force of 71.1 lb at the upper right corner in a direction perpendicular to the diagonal of the board. Suppose it has been determined that at the instant the lower edge of the board is horizontal, the center of gravity has an acceleration with a component of 25 ft per sec per sec toward the hinge and a component of 15 ft per sec per sec in the direction of the 71.1-lb force. Find the components, H and V, of force acting at the hinge at this instant. The arrows, upward on the V component and to the left on the H component, show the *assumed directions*, and these directions are taken as positive in writing the equations below.

Resolving horizontally,

$$H + 0.6 \times 71.1 = \frac{100}{g}(0.8 \times 25 + 0.6 \times 15) \qquad (1)$$

$$H = 90.1 - 42.7 = 47.4 \text{ lb.}$$

Resolving vertically,

$$V + 0.8 \times 71.1 - 100 = \frac{100}{g}(0.8 \times 15 - 0.6 \times 25) \qquad (2)$$

$$V = 100 - 56.88 - 9.33 = 33.8 \text{ lb.}$$

It is clear that had the direction of either H or V been assumed in the opposite direction, the answer for the component would have come out with a negative sign just as was the case in statics. Also, in writing a resolution equation, it is immaterial which direction along the axis is assumed positive, but the direction which is assumed positive must be positive for both the components of force and of acceleration, *i.e.*, components of force and of acceleration in the direction assumed positive are put into the equations with plus signs and the opposites carry negative signs.

Problems

1. A 60-lb mass is pulled along a horizontal plane where the coefficient of friction is 0.4 by a force of 40 lb at an angle of 30° with the horizontal. If the body starts from rest, how far does it move in the first 3 sec? *Ans.* 45 ft.
2. If the body of Prob. 1 is to be pushed along by a force downward at 30° with the horizontal, what force will be required for the same motion? *Ans.* 91.54 lb.
3. In Example 1, reverse the direction of the horizontal force and find its maximum value without causing N to become negative. With this force acting to pull the body down the plane, what will be the acceleration? *Ans.* $2g$ down the plane.
4. In Example 2, assume components of the reaction at the hinge in the directions of the given components of acceleration. Find these components and find the magnitude and direction of the hinge reaction. Show that it is the same as for the example.
5. A 50-lb wheel rolls down an inclined plane making an angle of 30° with the horizontal. If it is known that the friction force between the wheel and the plane is 10 lb, what is the acceleration of the center of gravity of the wheel? *Ans.* $0.3g$.
6. Find the angular acceleration and the angular velocity of the board of Example 2. [Start with the given normal and tangential accelerations and use Formula XXVIII or Eq. (3) of Art. 141 and Eq. (3) of Art. 144.]
   Ans. 6 rad per sec per sec; $\sqrt{10}$ rad per sec. The angular acceleration is counterclockwise. The direction of the angular velocity cannot be determined from the normal acceleration.
7. A rectangular plane 10 ft square has two edges horizontal with one at an elevation 6 ft above the other. A body projected along the plane parallel to the horizontal edges is later moving at an angle of 30° with these edges. If the coefficient of friction between the body and the plane is 0.5, what are the components of the acceleration of the body at this instant?
   Ans. $0.4g$ down the plane and $0.3464g$ parallel to the 10-ft edges.

8. If the speed of the body at the instant described in Prob. 7 is 8 ft per sec, what is the radius of curvature of its path at the point in question? [Find the component of acceleration normal to the path and use Eq. (3) of Art. 141.]
Ans. 38.3 ft.

9. A 40-lb solid cylinder has a cord attached to its curved surface and wound several times around the circumference. The cylinder is placed on a smooth horizontal plane and a horizontal force of 10 lb is applied to the end of the cord. Find the acceleration of the center. *Ans.* $0.25g$.
(The cylinder also turns, but that is not the present question.)

151. Constrained Motion. A force of constant magnitude may produce an acceleration of constant magnitude and variable direction, with velocities and displacements as expressed in Eqs. (3) and (6) of Art. 140, provided this force acts at every instant along the tangent to the path of the moving particle at that instant. In order that the direction may change, other forces must be developed which act at right angles to the direction of motion at every instant. This is the condition of *constrained motion* without friction.

A particle which is free to move only along a given line, such as a bead sliding on a wire, has *two constraints* and *one degree of freedom.* A body which is free to move in any direction on a given surface has *one constraint* and *two degrees of freedom.* A particle which is free to move in any direction in space has *no constraints* and *three degrees of freedom.* A car running on a track has *two constraints.* If the car jumps the track and runs on the ground, it has *one constraint.* A body attached to a cord, the other end of which is fastened to a fixed point, is free to move on the surface of a sphere and has *one constraint.*

When a particle running on a track changes the direction of its motion, the change in direction is caused by the reactions of the track. These reactions are the *constraining forces.* If there is no friction, these reactions are normal to the direction of the track and do not affect the speed of the particle. If the particle is driven by a constant tangential force (independent of the normal reactions), the effect of this force upon the magnitude of the velocity and upon the displacement measured along the path is exactly the same as if the motion were in a straight line.

A pin wheel is an example of constrained motion with a constant tangential force. The rim of the wheel is constrained to move in a circular path. The force exerted by the explosive is always tangent to the motion and is practically constant. The force exerted by the spokes on the rim is perpendicular to the motion.

There may also be one, two, or three constraints against angular motion, so that a body completely free in space has six degrees of freedom. *It should be noted that if the position of a body is to be defined by mathematical coordinates, there are required as many coordinates as there are degrees of freedom.* The position of the body in Prob. 7 of Art. 150, for example, could be given by two coordinates. This repeats what was said before. It is just as important in dynamics as in statics to describe a situation with the minimum possible number of variables. It is a general matter of mathematical efficiency.

152. Kinetic Energy. A force **P** acts on a mass m which is moving with a velocity v in the direction of the force. During a time interval dt, the displacement is $ds = v\,dt$, and the work done by the force **P** is

$$dU = P\,ds = Pv\,dt \tag{1}$$

The force **P** accelerates the mass m. The magnitude of the force in terms of the mass m and the acceleration **a** (which is equal to dv/dt) is given by Formula XXXII.

$$P = ma = m\frac{dv}{dt} \tag{2}$$

Substitution in Eq. (1) and integration give

$$dU = Pv\,dt = mv\,dv \tag{3}$$

$$U = \frac{mv^2}{2} + C_1 \tag{4}$$

If the integration is between the limits v_1 and v_2, Eq. (4) becomes

$$U = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} \tag{5}$$

If the body starts from rest, the expression for the work done in changing its velocity from zero to v is

$$U = \frac{mv^2}{2} \qquad \text{Formula XXXV}$$

In the British engineers' system, U is the work in foot-pounds, v is the velocity in feet per second, and m is the mass in slugs or geepounds. 1 slug = 32.174 pounds.

$$\text{Mass in slugs} = m = \frac{W}{g} = \frac{W}{32.174}$$

Formula XXXV may be written

$$U = \frac{Wv^2}{2g} \qquad \text{Formula XXXVI}$$

in which W is the mass in pounds and $g = 32.174$.

In the centimeter-gram-second system, m is mass in grams, v is velocity in centimeters per second, and U is work in ergs.

In the derivation of Formula XXXV, it is not necessary that the force P be constant. Formula XXXV gives the total work which is done on the mass m in bringing its velocity up from zero to v. It also gives the amount of work which the mass m can perform when its velocity changes from v to zero. This is called the *kinetic energy* of the mass m. Since v^2 is positive whether v is positive or negative, the kinetic energy is independent of the direction of the motion. Kinetic energy, like work, involves the product of two vectors along the same line and is, therefore, a scalar quantity.

Formula XXXVI expresses the kinetic energy of a body of which all particles have the same velocity. It is well for the student to think of v as the velocity of the mass center of the body. When a body has a rotational motion, the total kinetic energy is expressed by Formula XXXVI, with the velocity of its mass center used as v, plus an additional quantity which will be discussed later.

It should be noted that Formula XXXVI was derived from Formula XXXII and the definition of work. No new principle of mechanics has been added. It could therefore be presumed that any problem solvable by the use of Formula XXXVI could have been solved by Formula XXXII. This is true, but in many cases the use of Formula XXXVI leads to much simpler solutions.

The kinetic energy of a body is a scalar quantity, yet it is perfectly proper to deal with the kinetic energies of the rectangular components of a velocity separately when that is convenient. If v of Formula XXXVI is the velocity relative to any observer, the formula gives the kinetic energy of the body relative to that observer.

Example 1

A car weighing 3217 lb has its velocity changed from 60 ft per sec east to 20 ft per sec east while it moves 200 ft. Find the work done by the car. Find the retarding force in pounds.

Mass in slugs = 100.

$$\begin{aligned} U_1 - U_2 &= 50(3600 - 400) = 160{,}000 \text{ ft-lb} \\ P \times 200 &= 160{,}000 \\ P &= 800 \text{ lb} \end{aligned}$$

Problems

1. A mass of 60 lb has its velocity changed from 20 ft per sec east to 80 per sec east. What is the change in kinetic energy? If this change of velocity is caused by a constant force of 100 lb, how far does the body travel?

Ans. $U_2 - U_1 = \dfrac{60}{2 \times 32.174}(6400 - 400) = 180{,}000 \times 0.031081 = 5594$ ft-lb; $s = 55.94$ ft.

2. What is the kinetic energy of a cannon ball weighing 1000 lb and moving with a velocity of 2700 ft per sec? If this projectile is fired from a gun 50 ft in length, what is the average resultant force?

   Ans. $U = 113{,}290{,}000$ ft-lb; $P = 2{,}265{,}800$ lb.
3. If the shot of Prob. 2 strikes a target and is stopped in 5 ft, what is the average pressure which it exerts?
4. A car moving 30 ft per sec is stopped in 40 ft by four-wheel brakes. Assuming that the car is sliding on all wheels, what is the coefficient of friction?

   Ans. $\mu = 0.35$.
5. If a car running 15 mph can be stopped in 15 ft on a dry brick street, in what distance can this car be stopped when running 30 mph? 45 mph? Solve without writing.

In the example and problems above, the accelerating force is calculated by equating the change in kinetic energy with the work done on the body when the acceleration is positive, or with the work done by the body when the acceleration is negative. This is generally the better method when the *distance* traveled during the change of velocity is given. When the *time* required to change the velocity is specified, Formula XXXII is usually preferable.

Example 2

Solve Example 1 for the retarding force in pounds by means of Formula XXXII.

$$\text{Average velocity} = \frac{60 + 20}{2} = 40 \text{ ft per sec}$$

$$t = \frac{200}{40} = 5 \text{ sec}$$

$$a = \frac{20 - 60}{5} = -8 \text{ ft per sec per sec}$$

$$P = 100(-8) = -800 \text{ lb}$$

Problems

6. Solve Prob. 1 for the distance traveled by means of Formula XXXII.
7. A mass of 50 lb has its velocity changed from 70 ft per sec east to 10 ft per sec east while it moves 120 ft. Find the retarding force. Solve by work and energy. Check by force and acceleration.
8. A mass of 6 slugs has its velocity changed from 60 ft per sec east to 200 ft per sec east in 5 sec. Find the force in pounds. Solve by force and acceleration. Check by work and energy.

153. Potential Energy. Energy is capacity for doing work. *Kinetic energy* is the *energy of motion. Potential energy* is the *energy of position.* The work which the forces acting on a body do when the body moves from one position to another represents the differential energy for the two positions. For instance, a body of W pounds mass moves from position B of Fig. 288 to position A, which is h feet lower than B. Since the displacement from B to A is in the direction of the force of gravity, the work done by the attraction of the earth is Wh foot-pounds. The potential energy of the mass of W pounds is Wh foot-pounds greater at B than it is at A.

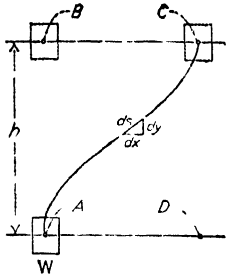

FIG 288.

The potential energy of the mass of W pounds is Wh foot-pounds greater at any point in the horizontal plane BC than it is at A, at a distance of h feet below this plane. If the body is moved from C to B, the gravitational force does no work, since the displacement is normal to the direction of the force. The body may move from C to A along any curve. For a small displacement ds along this curve, the component of the force in the direction of the displacement is $W \sin \theta$, in which θ is the angle which ds makes with the horizontal.

$$dU = W \sin \theta \, ds \tag{1}$$

Since

$$ds = \frac{dy}{\sin \theta}$$

$$dU = \frac{W \sin \theta \, dy}{\sin \theta} = W \, dy \tag{2}$$

$$U = W[y]_0^h = Wh \tag{3}$$

If there is no friction, the work depends upon the vertical distance of the two positions. This is merely a statement of the fact that work is the product of the force multiplied by the component of the displacement in the direction of the force (see Art. 9).

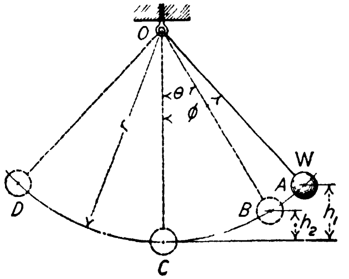

FIG. 289.

Besides the potential energy of gravitation, there are the energies of magnetic and electrical forces, and the potential energy of elastic forces, such as those of a bent spring or a cylinder of compressed gas.

When the potential energy of a body is reduced, this energy is changed into some other form. If none of the energy is dissipated as heat, light, or sound, or transferred as work on other bodies, all the lost *potential* energy goes to accelerate the body and thereby increases its *kinetic* energy. Figure 289 shows a mass of W pounds which is suspended from a fixed point O by a cord of length r. When the mass is displaced to a point A and then released, it swings down to the lowest point C and then up to a point D, which is practically as far from the vertical on the left as the point A is on the right. At A, the energy is all potential. The mass is at rest. At B, part of the potential energy has been changed into kinetic energy, and the mass has the velocity which it would acquire in falling a vertical distance $h_1 - h_2$. If the horizontal plane through the lowest point C is taken as the datum plane for the potential energy, the potential energy at A is

$$Wh_1 = Wr(1 - \cos\phi)$$

the potential energy at B is

$$Wh_2 = Wr(1 - \cos\theta)$$

and the potential energy at C is zero. At the lowest point C, all the potential energy has been changed into kinetic energy and the velocity has its maximum value. This kinetic energy takes the mass up on the other side until it stops at D, where all the energy again is potential.

If mass is given in slugs (geepounds) then weight in pounds is $mg = 32.174m$ and potential energy is mgh. In absolute systems, weight is equal to mg dynes or poundals, and potential energy is mgh.

When energy has been lost because of dissipative forces like friction, this energy must be accounted for in the energy equation. Such energy, of course, is not actually lost but is converted into heat or into wave motions like sound. The full equation sets up a balance of energy for the free body at two times t_1 and t_2 by stating: *Potential energy at t_1 plus kinetic energy at t_1 minus energy dissipated during the interval t_1 to t_2 equals the potential energy at t_2 plus the kinetic energy at t_2.* In mechanics the energy converted to heat is considered as lost. In thermodynamics the equation is a little more elaborate and contains the total heat energy of the free body as a separate item as well as the gains and losses of heat energy.

Example

A 40-lb body slides 20 ft down a 30° inclined plane, starting with an initial velocity of 8 ft per sec. If the first 6 ft of the plane was relatively smooth so that the friction force was only 5 lb while the remainder was rougher, offering a frictional resistance of 10 lb, what was the velocity of the body when it arrived at the bottom? Assume as the datum for potential energy a horizontal line through the bottom of the plane. Let v be the final velocity.

$$\frac{40 \times 8^2}{2g} + 40 \times 20 \sin 30^\circ - 6 \times 5 - 14 \times 10 = \frac{40v^2}{2g}$$

Problems

1. A mass of 64.35 lb is suspended from a cord as in Fig. 289. The center of gravity of the mass is 20 ft from fixed end of the cord. The body is displaced until the cord makes an angle of 60° with the vertical and then released. Using the lowest point as the reference position for potential energy, find the kinetic energy and the velocity when the cord makes an angle of 45° with the vertical.
Ans. $U_p = 376.96$ ft-lb; $U = 266.54$ ft-lb; $2v^2/2 = 266.54$; $v = 16.33$ ft per sec.
2. Solve Prob. 1 when the cord makes an angle of 32° with the vertical and also when the cord is vertical.
3. A mass of 20 lb slides 100 ft down a 30° smooth inclined plane. If the body started from rest at the top, what is the kinetic energy at the bottom, and what is the maximum velocity? *Ans.* $U = 1000$ ft-lb; $v = 56.72$ ft per sec.
4. Solve Prob. 3 if there is a friction force of 1 lb. *Ans.* $v = 53.81$ ft per sec.
5. Solve Prob. 3 if the plane is smooth and the mass has an initial velocity of 20 ft per sec down the plane. *Ans.* $v = 60.14$ ft per sec.
6. Solve Prob. 5 if the initial velocity is 20 ft per sec parallel to the plane in any direction.
7. Solve Prob. 4 if the body has an initial velocity of 20 ft per sec down the plane.
Ans. $U = 1000 + 124.32 - 100 = 1024.32$ ft-lb; $v = 57.41$ ft per sec.
8. Solve Prob. 4 if the body has an initial velocity of 20 ft per sec up the plane.
Ans. $U = 1000 + 124.32 - 122.60 = 1001.7$ ft-lb.
9. A body is thrown horizontally with a velocity of 40 ft per sec. What will be its velocity when it is 20 ft below the level of the starting point? What angle will its resultant velocity make with the horizontal?
10. A body falls from rest. Find its velocity after it has fallen h ft.
Ans $v^2 = 2gh$; $v = \sqrt{64.348h} = 8.02\sqrt{h}$.
11. A body is thrown with a velocity v_0 in any direction. What will be its velocity when it is a distance h below the initial position? Solve by work and energy.
Ans. $Wh + \dfrac{Wv_0^2}{2g} = \dfrac{Wv^2}{2g}$; $v^2 = v_0^2 + 2gh$.
12. Solve Prob. 10 of Art. 148 by work and energy for the two answers given.

154. Velocity in Terms of Displacement. For certain types of problems it is convenient to have equations expressing the relation of the initial and final velocities to the displacement and the acceleration. These equations are equivalent to energy equations with the forces and masses eliminated by use of the fundamental relation $P = Wa/g$. Suppose a body has an initial velocity v_0 and the resultant of all forces acting on the body in the direction of v_0 is P, then

$$\frac{Wv_0^2}{2g} + Ps = \frac{Wv^2}{2g}$$

where v is the final velocity and s is the displacement, both in the direction of the force P and the initial velocity. When the value of P

from Formula XXXIV is substituted in this equation, it reduces to

$$v^2 - v_0^2 = 2as \qquad (1)$$

When the initial velocity is zero, Eq. (1) becomes

$$v^2 = 2as \qquad (2)$$

When the final velocity is zero,

$$v_0^2 = -2as$$

For freely falling bodies, the acceleration is g. If both h and g are regarded as positive downward, Eq. (1) may be written

$$v^2 - v_0^2 = 2gh \qquad (3)$$

and Eq. (2) may be written

$$v^2 = 2gh \qquad v = \sqrt{2gh} \qquad \text{Formula XXXVII}$$

Formula XXXVII gives the velocity of a freely falling body after descending a distance h vertically. It is, of course, also the height to which a body will rise after it has been projected with a vertical (upward) component of velocity v, neglecting air resistance. If the body has also a horizontal component of velocity, this component remains constant except for air resistance. Only for relatively heavy projectiles at low velocities may air resistance be neglected without considerable error, but the problems involving air resistance are beyond the scope of this text.

Formula XXXVII and the other equations in this article are convenient for problems in which the accelerations are known rather than the masses of the bodies and the force systems. Such problems can be solved just as well by energy equations in most cases.

Example

A body is thrown upward with a velocity of 80 ft per sec. What will be its velocity when it is 20 ft above the starting point?

Using Eq. (3) with h positive and g negative,

$$v^2 = 6400 - 1287$$
$$v = 71.5 \text{ ft per sec}$$

Problems

1. In the example above there are two solutions for the velocity. Explain.
2. A body is thrown vertically upward with a velocity of 100 ft per sec. How high will it rise? In what time will it return to the starting point?

Ans. $h \doteq 155.4$ ft; time of ascent and return = 6.216 sec.

(Solve Prob. 3, 4, and 5 without writing. Use $g = 32$ and employ the average value of the velocity when convenient.)

3. A body is thrown upward with a velocity of 80 ft per sec. How long will it rise? What is the average velocity upward? How high will it go?

4. A body is thrown upward with a velocity of 80 ft per sec. What is its average velocity during the first second? How far does it go during the first second?

5. A body is thrown upward with a velocity of 40 ft per sec. What is its average velocity during the first second? Where will it be at the end of the first second? What is its average velocity during the first 2 seconds? Where will it be at the end of 2 sec? How far will it travel during the first 2 sec?

6. A body is thrown upward with a velocity of 160.87 ft per sec. What will be its velocity when it is 96.522 ft above the starting point?

7. A body is thrown upward with a velocity of 120 ft per sec. At what point will the velocity be 60 ft per sec? At what point will the velocity be 150 ft per sec?

8. In what time will a body fall 100 ft if it has an initial velocity of 60 ft per sec downward?

9. A body is projected horizontally with a speed of 40 ft per sec. What is its vertical velocity at the end of 2 sec? What is the direction and magnitude of its resultant velocity at the end of 2 sec? What is the vertical displacement at the end of 2 sec? What is the direction and magnitude of the resultant displacement at the end of 2 sec?

Ans. $v_y = -64.348$ ft per sec; $v = 75.79$ ft per sec at an angle of 58°08′ below the horizontal. Resultant displacement = 102.67 ft at 38°49′ below the horizontal.

10. Check the magnitude of the resultant velocity in Prob. 9 by work and energy.

11. A train is running over a trestle with a velocity of 40 ft per sec. A lump of coal falls from the tender and strikes the ground 30 ft below. Find the resultant velocity by work and energy. How long will it take for the coal to reach the ground? *Ans.* $v = 59.41$ ft per sec; $t = 1.366$ sec.

12. A train is running over a trestle with a velocity of 40 ft per sec. A lump of coal is thrown horizontally at right angles to the track with a velocity of 30 ft per sec. Find the magnitude and direction of the velocity after 2 sec.

Ans. $v = 81.5$ ft per sec at 52°10′ with the horizontal in a vertical plane 36°52′ with the direction of the track.

13. A body is thrown upward at an angle of 30° with the horizontal with a velocity of 150 ft per sec. Find the horizontal and vertical components after 3 sec.

Ans. $v_x = 129.90$ ft per sec; $v_y = -21.52$ ft per sec.

Figure 290 shows the path of a body which is projected with an initial velocity v_0 at an angle α with the horizontal. With y positive upward and g negative, the components of the velocity are

$$v_x = v_0 \cos \alpha$$
$$v_y = v_0 \sin \alpha - gt$$

The components of the displacement are

$$x = (v_0 \cos \alpha)t \tag{4}$$
$$y = (v_0 \sin \alpha)t - \frac{gt^2}{2} \tag{5}$$

When the time is given, Eqs. (4) and (5) may be used to solve the problem of position. When the time is not given, an equation of x in terms of y is better. When the value of t from Eq. (4) is substituted in Eq. (5), the result is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \tag{6}$$

The horizontal distance, OC of Fig. 290, from the initial position to the place at which the projectile is again at the initial level, is called the *range*. If $y = 0$ in Eq. (6),

$$x = 0 \text{ or } x = \frac{2v_0^2 \cos \alpha \sin \alpha}{g}$$

$$x = \frac{v_0^2 \sin 2\alpha}{g} \tag{7}$$

Figure 290 is drawn for an initial velocity of $5g$ feet per second. $\cos \alpha = 0.6$, $\sin \alpha = 0.8$ so that $v_x = 3g$ and the vertical component of the initial velocity is $4g$. In 4 seconds the vertical velocity is zero, and the projectile has reached its maximum altitude of $8g$ feet. In 4 more seconds the projectile is back at the original level. In the entire interval of 8 seconds the horizontal displacement is $24g$ feet.

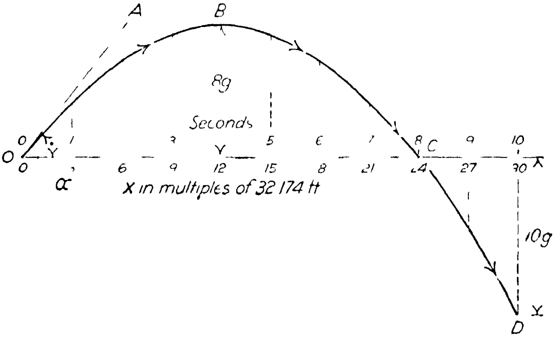

FIG. 290.

In Fig. 291, the first body is thrown at an angle with the horizontal and the second body is dropped at the same time from a point which is in the line of the initial velocity of the first body. The distance which the projectile falls from the line of its initial velocity is exactly the same as in the case when the initial velocity is horizontal or is zero. The projectile may be regarded as moving with constant speed along the line of its initial velocity and falling from that line with constant acceleration. If the paths of the projectile and of the falling body intersect, the bodies will collide. If the angle of projection is changed slightly, the vertical position of the projectile at every point on its path is changed just as much as the corresponding point on the line

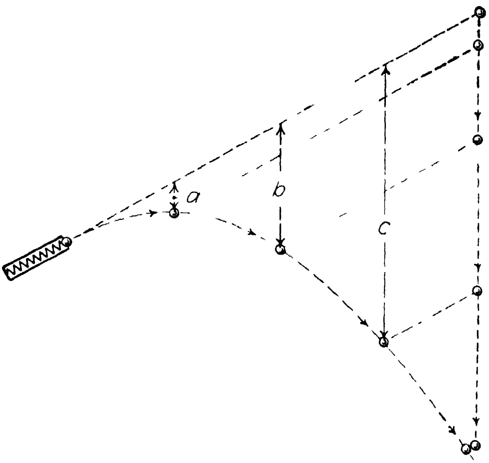

FIG. 291.

of projection was changed by the change in the angle. This is what is meant by the phrase "the rigidity of the trajectory."

Problems

14. What is the range of a projectile if the initial velocity is 100 ft per sec at an angle of 30° with the horizontal? *Ans.* 269.1 ft

15. What is the initial velocity required to throw a baseball 300 ft at the best angle, with no allowance for the air resistance?

16. A projectile is thrown horizontally. What must be its velocity in order to strike a target which is 100 ft lower than the initial point and 80 ft from the vertical line through it? *Ans.* $v_0 = 32.1$ ft per sec.

17. A projectile is thrown to pass through two rings. One ring is 80 ft higher than the initial point and 60 ft east of it. The other ring is 10 ft higher than the initial point and 100 ft east of it. Find the initial velocity and the initial angle. *Ans.* $v_0 = 76.2$ ft per sec at 72°34′ elevation.

18. A 16-lb shot leaves the athlete's hand 7 ft above the ground with a velocity of 36 ft per sec, at an angle of 45°. How far will it go horizontally. *Ans.* 46.36 ft.

19. Find the radius of curvature of the projectile of Prob. 14 at the instant when the velocity is horizontal. *Ans.* $7500/g$ ft.

20. Check Prob. 19 by writing the equation of the path of the projectile and using a formula for the radius of curvature. (The formula is in Art. 141.)

Energy equations are extremely useful in the mechanics of fluids. Formula XXXVII gives the velocity of the jet issuing from a large tank of relatively still liquid when h is the vertical depth of submergence of the orifice. Figure 292 represents a thin-lipped orifice in a tank with the liquid level h feet above the orifice. When a small amount, say 1 pound, of the liquid emerges from the orifice, the liquid moves down in the tank so that the small quantity appears to have been taken from the top. The missing potential energy reappears as kinetic energy of the pound of liquid at the contracted vein, and its velocity is $\sqrt{2gh}$. The term $\sqrt{2gh}$ in fluid mechanics is called *velocity head* and represents the kinetic energy per pound of flowing fluid. With a smooth orifice in a thin plate and a liquid of low viscosity like water, the ideal jet velocity is very nearly attained and the

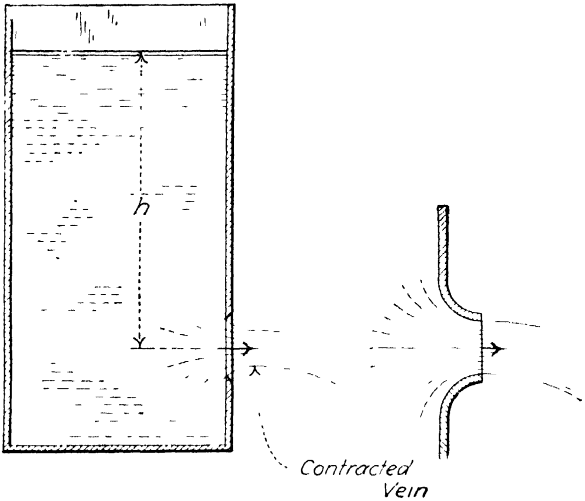

FIG. 292.

contracted vein will have an area of about 62 per cent of that of the orifice. If the outlet is rounded as shown in Fig. 292,II, the jet will be full-sized and the velocity will be about 99 per cent of the theoretical.

155. Connected Bodies. When the static equilibrium of connected bodies is considered, it is necessary only to take into account the mutual forces between the bodies. When the connected bodies have accelerations, it is necessary also to express the relations between the separate accelerations by equations consistent with the motion of the system. In statics problems, an over-all check of the solution of a system of connected bodies could readily be made by considering the equilibrium of the external forces. This procedure is correct in problems of dynamics, but the resolution equations must be based on the acceleration of the *center of gravity of the entire system* and the moment equations must include the angular acceleration of the entire system and the moment of inertia of the whole system of masses with respect to the center of gravity. Except for the simpler problems, the over-all check in dynamics becomes too cumbersome to be worth while.

In the example below, it is assumed that the cord does not stretch and it is obvious that any downward displacement of the 12-lb mass is accompanied by an equal displacement of the 20-lb mass to the right, *i.e.*, the *numerical values* of the displacements, velocities, and accelerations of the two bodies are equal at all times so long as the cord remains taut.

Example 1

Figure 293 shows a 20-lb mass on a smooth horizontal plane. A cord attached to the mass runs horizontally over a smooth pulley and down to a suspended mass

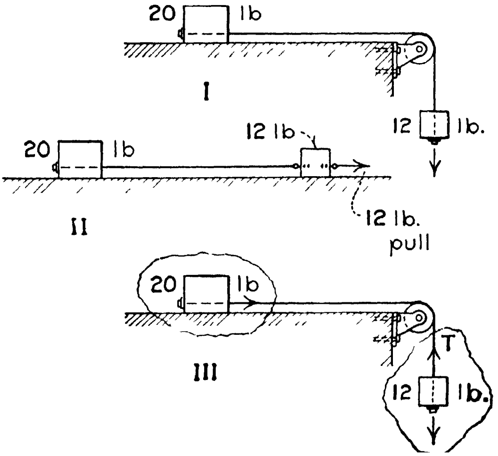

FIG. 293.

of 12 lb. Assume that the cord and the pulley have negligible masses and find the tension in the cord and the accelerations of the two masses.

Figure 293,III shows enclosing lines around the two masses to be treated as separate free bodies. Assume that the 12-lb mass is accelerated downward with an acceleration a and that the 20-lb mass has the same acceleration to the right. The effective force for the 12-lb mass is its weight minus the tension T. The pulley, in the absence of mass, requires no moment for its angular acceleration so the tension in the horizontal part of the cord is also T. (The 20-lb weight and the opposite normal reaction on the 20-lb mass are not shown in the figure.)

For the 12-lb mass, vertical resolution,

$$12 - T = \frac{12a}{g} \tag{1}$$

Horizontal resolution for the 20-lb mass

$$T = \frac{20a}{g} \tag{2}$$

Adding the two equations to eliminate T,

$$12 = \frac{32a}{g} \tag{3}$$

From Eq. (3), $a = 3g/8$ which, when substituted in either of the first two equations, gives $T = 7.5$ lb.

Figure 293,II shows the problem represented as an *equivalent system.* It is as though the two masses were placed on the smooth plane with the string taut and a 12-lb horizontal force applied to the 12-lb mass. A horizontal resolution on this system gives Eq. (3). (This is not the over-all check referred to in the opening paragraph. For this check, the force of the pulley on the string, 10.6-lb upward to the right at an angle of 45° with the horizontal, plus, vectorially, the weight of the 12-lb mass would have to be correlated with the acceleration of the mass center of the two bodies. This center has a component of acceleration to the right of $15g/64$ and a component downward equal to $9g/64$.) Equivalent systems are of great value to experts in dynamics, but they should not be used by the beginner until he has thoroughly mastered basic principles. Such systems are treated briefly in Chap. 19.

Problems

1. Solve the example above if the coefficient of friction between the 20-lb mass and the plane is 0.1.

Ans. $a = \frac{(12 - 2)}{32} g = 10.054$ ft per sec per sec; $T = 8.25$ lb.

2. Solve the example above if a force of 1 lb at the rim is required to overcome the friction of the pulley.

Ans. $a = 11g/32 = 11.060$ ft per sec per sec; vertical tension = 7.875 lb; horizontal tension = 6.875 lb.

3. A 40-lb mass is placed on a plane which makes an angle of 30° with the horizontal. The mass is attached to a cord which runs up parallel to the plane, passes over a smooth, weightless pulley, and supports a 36-lb mass on the free end. Find the acceleration and the tension on the cord if the plane is smooth. Check. *Ans.* $a = 4g/19 = 6.77$ ft per sec per sec. $T = 28.42$ lb.

4. Solve Prob. 3 if there is a coefficient of friction of 0.1 between the plane and the 40-lb mass. Check. *Ans.* $T = 30.06$ lb.

5. A mass of 50 lb is placed on a 30° inclined plane and attached to a cord which runs up the plane, passes over a smooth, weightless pulley, and supports a mass of 10 lb. Starting from rest, what distance will the 50-lb mass slide down the plane in 3 sec if the plane is smooth?

6. If the system of Prob. 5 moves 27 ft in 3 sec from rest, what is the coefficient of friction between the 50-lb mass and the plane?

Ans. Friction = 3.81 lb; $\mu = 0.088$.

7. Solve Prob. 6 by work and energy. *Ans.* Friction = 102.89/27 = 3.81 lb.

8. A 20-lb mass is placed on a plane which makes an angle of 20° with the horizontal. The mass is attached to a cord which runs up the plane, passes over a smooth, weightless pulley, and supports a mass of 12 lb on the free end. Starting from rest, the system moves 18 ft during the first 3 sec. Find the coefficient of friction between the plane and the 20-lb mass. *Ans.* $\mu = 0.063$.
9. Check Prob. 8 by work and energy.
10. Figure 294 shows a cord which runs over a pulley and supports a mass of 6 lb on one end and a mass of 4 lb on the other. If the pulley is weightless and frictionless and if the cord is weightless, what velocity will the system acquire in 4 sec after starting from rest? What distance will it travel in the 4 sec? What is the tension in the cord? What is the total load on the pulley? If this load is less than 10 lb, explain the discrepancy?

Figure 294 represents the Atwood machine, which is sometimes used to demonstrate the laws of acceleration and to determine g approximately. The machine is subject to errors which result from the friction and from the mass of the pulley. These errors may be determined experimentally and corrections made. Error caused by the varying length of cord on the two sides may be avoided by having the cord continuous as is shown in the figure. The total mass of the cord must be added to the mass of the two suspended bodies to get the mass which is accelerated.

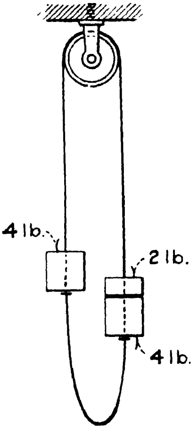

FIG. 294.

Example 2

Figure 295 shows a 60-lb mass on a horizontal plane where the coefficient of friction is 0.25, a two-pulley system, and a pendent mass of 40 lb. Assume both pulleys and the cord to be of negligible mass and the pulleys to be frictionless. Find the tensions in the cord and the accelerations of the two masses. From the velocity ratio, the acceleration of the 60-lb mass is one-half that of the 40-lb mass. Let a equal the acceleration of the 60-lb mass. Calling the downward acceleration of the 40-lb mass $2a$, the equations for the two unknowns, T and a, are

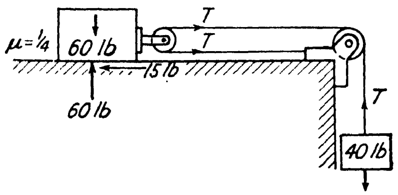

FIG. 295.

$$40 - T = \frac{40(2a)}{g} \tag{1}$$

$$2T - 15 = \frac{60a}{g} \tag{2}$$

Problems

11. Solve Example 2 for the accelerations and tensions while the bodies are coming to rest after the 60-lb body has been given an initial velocity to the left and the 40-lb body a corresponding velocity upward.

12. Neglect the mass and friction of the pulleys of Fig. 296 and the mass and stiffness of the rope, and find the tension in the rope and the accelerations of the bodies when $W = 160$ lb and $P = 100$ lb. *Ans.* $T = 85.71$ lb.

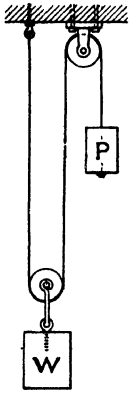

Fig. 296.

156. Miscellaneous Problems

1. A mass of 100 lb is moving on a horizontal plane with a velocity of 60 ft per sec. What is the coefficient of friction required to stop the body in 150 ft? Solve by work and energy and check by force and acceleration.

2. Solve and check Prob. 1 if the mass is brought to rest in 5 sec. Which is the better method when time is given? Which is the better method when displacement is given?

3. An inclined plane 50 ft long has its lower end at the top of a wall which is 40 ft high. The plane makes an angle of 45° with the horizontal. When a body slides down from the top of the plane, how far from the wall will it strike the ground? Solve for a smooth plane and also for a coefficient of friction of 0.2 *Ans.* $x = 28.51$ ft; $x = 27.05$ ft.

4. A body projected with a velocity of 24 ft per sec northward on a smooth sheet of ice crosses the east-west line from which it was projected 6 sec later at a point 108 ft east of the starting point. Assume that the change of velocity was due entirely to a wind acting on a 32.16-lb mass and find the magnitude and direction of the wind force and the net work done by the wind on the body. *Ans.* 10 lb south 36°52′ east; work done 648 ft-lb.

5. A ball is thrown with a velocity of 61.32 ft per sec at an angle of 30° with the horizontal. What will be the location of center of curvature of its trajectory 1 sec later? *Ans.* 55.6 ft horizontally and 80 ft below the point of release.

6. A cord passed over a fixed cylinder hangs vertically on both sides. A 12-lb mass is attached to one end and a 4-lb mass to the other. It is found that the 12-lb mass moved downward uniformly after being started. What acceleration is to be expected if 2 lb is removed from the 4-lb mass and added to the 12-lb mass? *Ans.* $g/3$.

7. Solve Example 1 of Art. 155 if the pulley is replaced by a fixed cylinder with a coefficient of friction of 0.3 between the cord and the cylinder.

8. In Fig. 200, neglect the mass of the wheel and neglect friction and find the accelerations and tensions when $P = 10$ lb, $W = 40$ lb, $a = 3$ in., and $r = 24$ in. *Ans.* Mass P accelerates at $8g/17$ ft per sec per sec.

9. Neglect friction and rotating mass and find the accelerations and tensions in the cords of Fig. 201 when $P = 10$ lb, $W = 100$ lb, $a_1 = 3$ in., $a_2 = 4$ in., and $r = 20$ in.

10. A small car running without friction down a 30° inclined plane has a horizontal top upon which a load rides down the plane. Find the necessary coefficient of friction between the car and the load to keep the load in place during the motion. *Ans.* 0.577.

11. A V-shaped trough with 90° between the sides is used as a conveyor down which concrete cylinders slide. How long will be required for a cylinder to slide 10 ft from rest when the trough is inclined at 30° with the horizontal if the coefficient of friction is 0.2?

12. Sand grains of the same size and shape fall from a sieve which is 2 ft in diameter. Under the sieve a fan blast is adjusted to give the lighter grains an acceleration of 8 ft per sec per sec. The heavier grains are 1.6 times as dense as the lighter. How far below and down-wind from the center of the sieve must a separator be placed to divide the grains into two homogeneous piles? (Neglect occlusion and other minor effects.) *Ans.* $x = 4.33$ ft; $y = 21.33$ ft.

13. A 40-lb body on light smooth rollers runs down an inclined plane of slope 3 vertical on 4 horizontal. It drags behind it a 150-lb body under which the coefficient of friction is $\frac{1}{3}$. If the cord connecting the two bodies is horizontal, what tension does it sustain?

14. A body starts from rest and slides down a 30° inclined plane. For the first 36 ft the plane is relatively smooth and offers a friction resistance of 6 lb. The remainder of the plane is very rough and the friction force is 16 lb. Find the distance to stop and the time for the complete motion if the body weighs 24 lb.

15. A body starts from rest, slides 18 ft down a smooth 30° inclined roof, and falls to the ground 12 ft below. Find the horizontal distance from the edge of the roof to the striking point. *Ans.* 11.9 ft.

16. In Fig. 293, suppose the cord from the 20-lb mass is not horizontal but the pulley is raised so that this cord makes an angle of 30° with the horizontal at an instant when the 20-lb mass has a velocity of 8 ft per sec. Find the tension in the cord and the accelerations of the two bodies at this instant. (The two accelerations are not equal. The proper relation between them can be found by calculus.)

CHAPTER 16

DYNAMICS OF ROTATION

157. Torque and Angular Acceleration. When a rigid body has an angular motion, every particle of that body at any instant during the motion has an acceleration, and this acceleration is produced by forces applied to the particle by its neighbors in accordance with Formula XXXIV. The external force system acting on the body, therefore, must be resolvable into a large number of components so that there exists a component to account for the acceleration of each part of the body. The actual external force system may be relatively simple with forces applied to the body at only a few points, but a system of internal stresses is induced in the body so that each particle is subjected to the force required for it to remain in its position relative to the other particles of the body. (Of course if the internal stress at any point or points exceeds the rupture strength of the material, the body breaks into two or more pieces and ceases to behave as a single rigid body.)

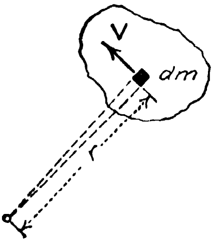

Fig. 297.

Figure 297 represents a particle of mass *dm* of a rigid body rotating about the center *O*. The tangential velocity of the particle is *V* and the acceleration of the particle in the tangential direction is $r\alpha$, where α is the angular acceleration of the whole rigid body. Assuming the external force system acting on the body to be resolved into the components required to account for the accelerations of the various particles of the body, there is a component at this element of mass, in the tangential direction,

$$dP = r\alpha\, dm \tag{1}$$

The moment of this component of force with respect to *O* is

$$dT = r^2\alpha\, dm \tag{2}$$

The total moment required to produce an angular acceleration α in a body made up of elements *dm* is

$$T = \alpha \int r^2\, dm = I\alpha \qquad \text{Formula XXXVIII}$$

In this formula T represents the *torque*, which is the name usually given to moment about an axis of revolution. The letter I of the formula represents the moment of inertia of the body with respect to the axis about which the moment T is taken. The formula applies as written to the cgs system and to the foot-poundal-second system. It applies also to the British engineers' system if m is expressed in geepounds ft². If k is the radius of gyration of the body in feet and W is its mass in pounds, Formula XXXVIII becomes

$$T = \frac{Wk^2\alpha}{g} \qquad \text{Formula XXXIX}$$

The product Wk^2 of Formula XXXIX is the moment of inertia of the body in pounds feet². If this factor is represented by I, the formula becomes

$$T = \frac{I\alpha}{g} \qquad \text{Formula XL}$$

With units of feet, seconds, and pounds (force), the only difference between the proper usage of Formulas XXXVIII and XL is that in using Formula XXXVIII the moment of inertia in pounds feet² is divided by g *before* inserting it into the formula whereas, in using Formula XL, dividing by g is considered part of the operation of using the formula. Formula XL is the better one to memorize, as it constantly keeps reminding the student not to leave out the g.

Example 1

A wheel 4 ft in diameter weighs 322 lb. The radius of gyration is 1.8 ft. What torque in foot-pounds will give the rim of this wheel a velocity of 24 ft per sec from rest in 4 sec?

Linear acceleration = 6 ft per sec per sec. Angular acceleration = 3 rad per sec per sec. $I = 10 \times 1.8^2 = 32.4$ approximately, geepounds feet².

$$\text{Torque} = 32.4 \times 3 = 97.2 \text{ lb-ft}$$

Problems

1. A homogeneous solid cylinder is 4 ft in diameter and weighs 240 lb. A rope to which a pull of 60 lb is applied is wound several times around the cylinder and fastened to it. What is the torque, if the diameter of the rope is neglected? What is the angular acceleration? What is the angular velocity 3 sec after it is 32 rad per sec? What is the angle traveled in the 3 sec?
Ans. $T = 120$ ft-lb; $\alpha = 8.044$ radians per sec per sec; $\omega = 56.132$ rad per sec; $\theta = 132.198$ rad.

2. A pulley 5 ft in diameter weighs 300 lb and has a radius of gyration of 2 ft. When the pulley is running 240 rpm, a pressure of 250 lb is applied to the rim by a brake. The coefficient of friction is 0.2. Find the negative acceleration, the time required to stop the pulley, and the number of revolutions traveled.

   Ans. $t = 7.499$ sec; $\theta = 14.998$ rev.

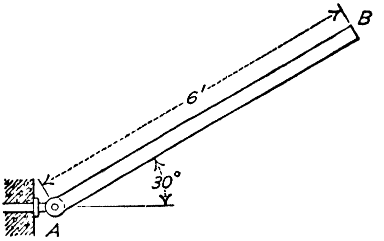

FIG. 298.

3. A uniform bar AB (Fig. 298) is 6 ft long, weighs 12 lb, and is hinged at the left end A. What is the moment about the hinge when the bar makes an angle of 30° with the horizontal? What is the angular acceleration?

   Ans. $\alpha = 6.966$ rad per sec per sec.

4. Solve Prob. 3 for the horizontal position of the bar.

   Ans. $\alpha = 8.044$ rad per sec per sec.

Example 2

Solve Prob. 1 if a mass of 60 lb is hung on the free end of the rope, as shown in Fig. 299.

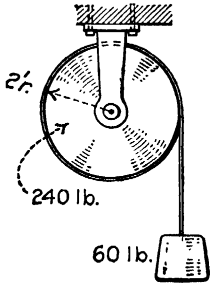

FIG. 299.

Since some of the pull of gravity on the 60-lb mass is required to accelerate the mass, the tension in the rope is less than 60 lb. If P represents the tension in the rope, the effective force in the 60-lb mass is $60 - P$. When this mass is treated as a free body with linear acceleration a, Formula XXXIV gives

$$60 - P = ma = \frac{60}{g} a \qquad (1)$$

The torque on the cylinder is $2P$. From Formula XL

$$2P = \frac{I\alpha}{g} = \frac{240 \times 4}{g \times 2} \alpha = \frac{480}{g} \alpha \qquad (2)$$

Substituting $a = 2\alpha$ in Eq. (3) and eliminating P between Eqs. (3) and (4),

$$\alpha = \frac{g}{6} = 5.362 \text{ rad per sec per sec}$$

$$P = 40 \text{ lb}$$

Problems

5. A pulley 5 ft in diameter weighs 875 lb. Its radius of gyration is 2 ft. A rope wound around the rim carries a mass of 240 lb on the free end. Find the acceleration and the tension on the rope if there is no friction.

   Ans. Tension = 168 lb; $\alpha = 0.12g$ radians per sec per sec.

6. A brake pushes against the rim of the pulley of Prob. 5 with a force of 200 lb. The coefficient of friction is 0.2. Find the acceleration and the tension in the rope.

   Ans. $T = 180$ lb; $\alpha = g/10$.

158. Moment Equations of Dynamics. Formula XL is of the utmost importance in applied mechanics. Without its use, every problem involving accelerated angular motion of sizable masses requires the use of integral calculus for its solution. With the formula, the integral elements of the problem are incorporated in the moment of inertia of the mass of the body.

The formula is a moment formula exactly analogous to the moment equation of statics. In statics, the sum of the moments of all the forces of the system acting on the free body with respect to any chosen axis is equated to zero. In dynamics the moment is computed in exactly the same way but instead of being equated to zero it is equated to the product of the moment of inertia (in proper units) of the body with respect to the axis of moments and the angular acceleration of the body. Of course, if the angular acceleration happens to be zero, the equation is exactly the same as for statics.

There is another difference between the equations of moments in statics and dynamics, and a highly important one. In statics, any axis could be used. In plane dynamics there are *never more than two* axes for which Formula XL (in any of its various forms) may be used. The derivation in Art. 157 was for an axis of rotation, *i.e.*, point O was a fixed center or an instantaneous center. The formula is also properly used when the axis is through the mass center, whether or not this is the center of rotation.[1] Figure 300 represents a body which is free to rotate in the plane of the paper about an axis through O. The center of mass is located at C at a distance $\bar{r}$ from the axis. A force **P** parallel to the plane of the paper intersects the plane through the axis and the center of mass at a distance r' from O. The angle between the force **P** and the line OC is ϕ. The force **P** is the resultant of all the forces (except the hinge reaction) which act on the body parallel to the plane of the paper.

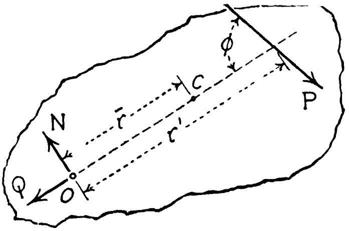

FIG. 300.

Let m equal the mass of the body and I_c its moment of inertia with respect to its mass center. The hinge reaction is represented by the component Q in the direction C to O and the component N perpendicular to CO. The moment of **P** with respect to O is $Pr' \sin \phi$. The moment of inertia of the body with respect to the axis through O is $I_c + m\bar{r}^2$. Substituting these values in Formula XL, there results

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[1] When the distance from the mass center to the instantaneous center is a variable, use of the formula with an axis through the instantaneous center is not correct except for small oscillations. Proof of this proposition can be found in more advanced texts.

$$Pr' \sin \phi = \frac{(I_c + m\bar{r}^2)\alpha}{g} \tag{1}$$

Solving for α,

$$\alpha = \frac{g(Pr' \sin \phi)}{I_c + m\bar{r}^2} \tag{2}$$

The tangential acceleration of the center of mass C is

$$\bar{r}\alpha = \frac{g(Pr' \sin \phi)\bar{r}}{(I_c + m\bar{r}^2)} \tag{3}$$

(This acceleration is in a direction opposite to the arrow shown for N.) Resolving on an axis perpendicular to OC (with the direction of the acceleration assumed positive),

$$P \sin \phi - N = \frac{mg(Pr' \sin \phi)\bar{r}}{g(I_c + m\bar{r}^2)} \tag{4}$$

From Eq. (4),

$$N = P \sin \phi - \frac{(P \sin \phi)m\bar{r}r'}{(I_c + m\bar{r}^2)} \tag{5}$$

The moment of the force **P** of Fig. 300 about the mass center C is $(P \sin \phi)\ (r' - \bar{r})$ and the moment of N is $N\bar{r}$, both clockwise as seen in the figure. The moment of Q about this axis is zero, so that the moment of the system is

$$N\bar{r} + (P \sin \phi)(r' - \bar{r}) = \frac{I_c\alpha'}{g} \tag{6}$$

When the value of N from Eq. (5) is substituted in Eq. (6), α' is found to be the same as α of Eq. (2).

The magnitude of the angular acceleration is the *same for either center* and Formula XL gives a correct equation regardless of which of the two centers is used. (To an observer moving along with any point on a body but not rotating with it, that point seems to be a stationary center and the body appears to have the same angular velocity and the same angular acceleration regardless of which point is so observed.) Formula XL is the formula for writing moment equations for free bodies which are acted on by force systems not in equilibrium. For such systems, *the axis of moments must be either through the mass center of the body or through the (instantaneous or fixed) center of rotation.* If the body has zero angular acceleration, the right-hand member of the formula is zero, but only the mass center axis may be used since the center of rotation is at infinity.

The student should not consider any of the equations above to be formulas to be remembered and to be used merely by substituting data and solving It is usually more trouble to reduce the system of forces acting on the body (exclusive of the hinge reaction) to the single force **P** than it is to carry out the idea involved: Take moments about *either the center of rotation or the center of gravity* (according to convenience) as in statics, and set the moment equal to the moment of inertia of the body (with respect to the center used) multiplied by its angular acceleration. *Never use any center other than one of these two. Never use an instantaneous center if its distance from the center of gravity is changing.*

Example 1

Figure 301 shows an 800-lb car being pulled along a horizontal plane by a 200-lb horizontal force 3 ft above the plane. The wheel base is 4 ft and the center of mass is 2 ft above the plane and midway between the axles. Assume the wheels to be of negligible mass and neglect all friction and rolling resistance. Find the vertical reactions under the axles.

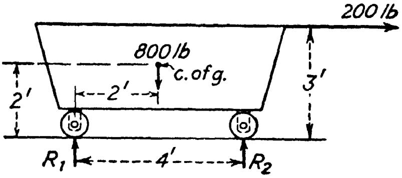

FIG. 301.

It is *assumed* that the wheels remain in contact with the plane, *i.e.*, (1) the center of mass moves in a straight horizontal line and has no acceleration vertically, and (2) there is no angular motion so the angular acceleration is zero. (These assumptions here seem simple and obvious, but it is important that the student understand that they are assumptions. If R_1 and R_2 turn out to be positive, the assumptions are justified. If either of them is negative, then either (*a*) the solution is in error or (*b*) some additional restraint must be provided, such as an upper rail to hold the wheel down or a magnetized rail for it to run on.)

There are three unknowns, R_1, R_2, and a, the linear acceleration of the center of gravity (assumed to the right).

The three independent equations are

A horizontal resolution

$$200 = \frac{800a}{g} \tag{1}$$

A vertical resolution,

$$R_1 + R_2 - 800 = 0 \tag{2}$$

Moments about mass center,

$$200 \times 1 + 2R_2 - 2R_1 = \frac{I_c\alpha}{g} = 0 \tag{3}$$

Solving the equations, $a = \frac{1}{4}g$, $R_1 = 350$ lb, and $R_2 = 450$ lb.

The type of problem presented above is often found in texts with a solution depending on the device of adding to the actual system of forces a fictitious force "acting" through the center of gravity and equal numerically to the resultant of the actual force system but in the opposite direction. This fictitious force is variously called the *inertia force,* the *reversed effective force,* the *force of acceleration, D'Alembert's force,* etc., and the force system modified by the addition of the fictitious force is sometimes called *dynamic equilibrium.* The procedure is somewhat illogical and perhaps a little mystical. It yields correct answers for unknowns such as R_1 and R_2 of the example above but gives zero for the acceleration. The procedure is not difficult to carry out in cases where the body has no angular acceleration, but when there is angular acceleration the fictitious force must not be placed at the center of gravity but must pass through a different point or must be accompanied by an additional fiction which may be called the *reversed effective torque.* Finding the proper location of the fictitious force or computing the concomitant fictitious torque is usually more laborious than is a correct, logical solution of the problem. The addition of the fictitious force can be logically justified only by assuming an accelerating frame of reference. This procedure is discussed briefly in a later chapter. The student is well advised to adhere strictly to the basic principles and to reserve the use of fictitious forces until he comes to problems involving kinematics so difficult that the use of moving frames of reference becomes advisable.

Problems

1. Assume the rear wheels of the car of Example 1 to be locked and to slide along the plane with a coefficient of friction of 0.5; find the acceleration while the car is being pulled to the right by the 200-lb force.
Ans. $R_1 = 280$ lb; $a = 0.075g$ to the right.

2. Solve Prob. 1 if the front wheels are locked and the rear wheels run freely, assuming the same coefficient of friction, the same horizontal pull, and motion still to the right. *Ans.* $R_2 = 600$ lb; $a = 0.125g$ to the left.

3. Assume the conditions of Prob. 1 and find the distance required for the body to come to rest if the 200-lb force is removed when the body is moving to the right with a velocity of 35.2 ft per sec. *Ans.* 96.28 ft.

4. A body moving down a plane with a slope of 3 vertically on 4 horizontally is provided with a slider at the front and a frictionless roller at the rear. The center of mass is 2 ft from the plane and on a line bisecting the 3-ft distance between the slider and the roller. Find the acceleration of the body as it moves down the plane if the coefficient of friction is 0.6. *Ans.* $0.36g$.

5. Solve Prob. 4 with the positions of the slider and roller reversed.

6. Solve Prob. 4 if the body is given an initial velocity up the plane.

7. The body of Prob. 4 is placed on a horizontal plane with the slider at the right. A horizontal cord is attached to the body 1 ft above the plane and extended to the right over a light, smooth pulley and down to a suspended body with a mass one-half that of the body on the plane. Find the tension in the cord when the suspended mass is moving downward, assuming a coefficient of friction of 0.6 between the slider and the plane. (There are four unknowns, the tension in the cord, the acceleration, which is the same for both bodies, and the two reactions. Three independent equations are available from the body on the plane and one from the suspended body.)

Ans. $T = 0.425W$, W being the mass on the plane.

8. Solve Prob. 7 when the body on the plane is slowing down after having been given an initial velocity to the left.

Example 2

Figure 302 represents a wheel rolling down an inclined plane. V indicates the velocity of the center of the wheel and B the point of contact with the plane. B is also the instantaneous center if there is no slipping. The curved arrow is the angular velocity, with angular acceleration in the same sense. Find the components of the reaction at B if the wheel weighs 100 lb, has a radius of 2 ft, and a radius of gyration with respect to its center of 1.5 ft, when it is on a 30° inclined plane where the coefficient of friction is 0.3 and where the rolling resistance is negligible.

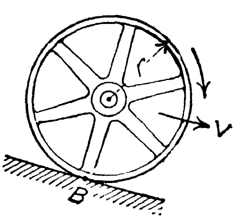

FIG. 302.

The unknowns are the magnitude and direction of the force acting on the wheel at B, the linear acceleration of the center of the wheel down the plane, the angular acceleration, the linear velocity of the center of the wheel, and the angular velocity—a total of six. The linear velocity of the wheel down the plane, while a function of the former acceleration and the time, is unrelated to the linear acceleration in this direction at this instant. The angular velocity of the wheel is related to the accelerations of all parts of the wheel *except the center*. The linear and angular velocity, under the assumption to be made, may be neglected. The answers to the problem would be the same whether the wheel were rolling up the plane or down, or whether it had just been released and had acquired no velocity.

Assume that there is no slipping between the wheel and the plane at B. Under this assumption B is an instantaneous center but the friction force at B is not necessarily the normal force multiplied by the coefficient of friction.

Let W = mass of the wheel in pounds, also its weight (known),
a = acceleration of the center down the plane,
α = angular acceleration of the wheel,
N = normal force at B acting toward the center of the wheel,
F = friction force at B acting (on the wheel) up the plane.

There are available for solution two independent equations of resolution and one of moments. The second equation of moments is not independent but may be used as a check on the work. With the center assumed at O, there is also the kinematics equation $a = r\alpha$.

Moment of inertia of the wheel about the center: $mk^2 = 225$ lb-ft²

Moment of inertia about B: $I_b = I_c + md^2 = 225 + 400 = 625$ lb-ft²

Taking moments about point B,

$$Wr \sin 30° = \frac{I_b\alpha}{g} \qquad 100 = \frac{625\alpha}{g} \tag{1}$$

From Eq. (1),

$$\alpha = \frac{4g}{25} \qquad a = r\alpha \qquad a = 2\alpha \qquad a = \frac{8g}{25} \tag{2}$$

Resolving parallel to the plane,

$$W \sin 30° - F = \frac{Wa}{g} \qquad 50 - F = \frac{100}{g} \times \frac{8g}{25} \tag{3}$$

From Eq. (3),

$$F = 18 \text{ lb}$$

Resolving normal to the plane,

$$W \cos 30° - N = 0 \qquad N = 86.6 \text{ lb} \tag{4}$$

Since $F < 0.3N$, the assumption of no slipping is justified. Taking moments about the center of mass for a check,

$$Fr = \frac{I_c\alpha}{g} \qquad 18 \times 2 = \frac{225}{g} \times \frac{4g}{25} \text{ Check} \tag{5}$$

Problems

9. Solve Example 2 if the coefficient of rolling friction between the plane and the wheel is 0.02 ft. (In this case N does not act at B but through a point 0.02 ft in the direction of rolling from B. Equation (4) should be written first, and Eq. (1) becomes $Wr \sin 30° - 1.73 = I_b\alpha/g$. The answer for a is slightly less and for F slightly more than before.)

10. Solve Prob. 9, assuming the wheel to be rolling up the plane.

11. An empty oil drum 1.5 ft in diameter with a weight of 40 lb and a radius of gyration of 0.707 ft (with respect to the center) is blown along a horizontal surface by a wind. Neglect rolling resistance and assume the wind force as 30 lb acting upward at 20° with the horizontal and along a line below and 0.2 ft from the central axis of the drum. Find the acceleration and find the components of force exerted on the drum by the horizontal surface.

12. In Fig. 207, assume the larger diameter as 4 ft, the smaller diameter as 1.5 ft, W as 400 lb and the radius of gyration of the reel (and flanges) as 1 ft with respect to the center. Neglect the rolling resistance and find P to give the center an acceleration of 4 ft per sec per sec.

13. Solve Prob. 12 with the reel on an inclined plane as shown in Fig. 192.

14. In Fig. 145, assume the 400-lb body to be a solid homogeneous cylinder, neglect friction and inertia of the pulleys, and find W to give the cylinder an acceleration of $0.25g$ up the plane.

In Example 2, the friction force at the point of contact was 18 lb and the normal force at this point was 86.6 lb. This is possible if the coefficient of friction is greater than 18/86.6. With a small coefficient of friction, the friction force could be only 86.6 times the coefficient and there would be some slipping at the point of contact. The resultant force down the plane would be $100 \sin 30° - 86.6\mu = 100a/g$. The moment about the center of the wheel would be the friction force times the radius of the wheel, and the angular acceleration can be found from this. The instantaneous center is not at the point of contact under this condition but is below the plane at a distance which can be determined by the equation $a = r\alpha$, where r is the distance from the instantaneous center to the center of wheel.

Problems

15. Solve Example 2 if the coefficient of friction between the wheel and the plane is 0.15.

16. Locate the instantaneous center for the wheel in Prob. 15, and check the problem by moments about this center.

17. In Prob. 12, find the necessary coefficient of friction for true rolling.

18. Solve Prob. 12 if the cable comes off at the bottom of the drum and find the coefficient of friction required for true rolling.

159. Centrifugal Force. *When a particle rotates about an axis, the acceleration toward the axis which results from the change of direction is*

$$a = \frac{v^2}{r} = r\omega^2$$

The force which causes acceleration is ma, and the force which causes this acceleration in a mass dm is $(v^2 dm)/r$, or $r\omega^2\, dm$. The force on the particle which causes change in direction is toward the axis. Regarding the particle as the free body, the applied force is *centripetal.* The force which the rotating particle exerts on the body or surface which constrains it to follow the curved path is directed away from the axis. This reaction, therefore, is *centrifugal.*

The word *centrifugal* is compounded of two Latin words meaning to fly away from the center, and the word *centripetal* from two Latin words meaning to put (or push) toward the center. In the literature of mechanics, these words have not always been employed with clear, consistent, and unambiguous meanings; but when applied to the forces acting on a *particle,* they nearly always have the meanings indicated above. When applied to a sizable body, the word *centripetal* is usually taken to mean the sum of the components of all forces acting on the body in the direction from center of gravity to center of turning, and this is the preferred meaning. Referring to Fig. 300, the centripetal force for the system shown is $Q - P \cos \phi$, and the centrifugal force is

$P \cos \phi - Q$. Neither, in this case, is a single definite force. If the force P in this figure were zero, the force Q which the pin at O exerts on the body would be the centripetal force and the equal and opposite reaction of the body on the pin at this point would be the centrifugal force.

When the frame of reference used for the study of a problem in dynamics is rotating with uniform velocity, some difficulties of kinematics are avoided by applying to the body a fictitious outward (from the center of rotation *of the frame*) force component. This component is nearly always called the *centrifugal force*. In the simplest type of rotational problem, all parts of the body under study rotate about an axis with uniform angular velocity. If a rotating frame of reference for this body is adopted with its axis coinciding with the axis of rotation and angular velocity of the frame equal to the angular velocity of the body, the body is at rest with respect to the frame. With the addition of a proper set of centrifugal forces to the body, the problem becomes a problem of statics. If the student expects ever to solve anything beyond this simplest type of problem, he should refrain from using the fictitious centrifugal force and should omit this "force" from his free-body diagrams. The diagrams then show no forces except gravity and the contact forces exerted by other bodies.

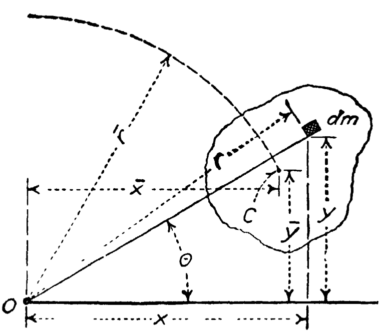

FIG. 303.

Figure 303 represents a body rotating about an axis which passes through O and which is perpendicular to the plane of the paper. An element of mass dm is subjected to a force which keeps it in its path around O. This force is directed toward O and is

$$dP = r\omega^2 \, dm \qquad (1)$$

A resolution parallel to the X axis of these infinitestimal components acting on the whole body gives

$$dP_x = \omega^2 r \cos \theta \, dm = \omega^2 x \, dm \qquad (2)$$

$$P_x = \omega^2 \int x \, dm = \omega^2 m\bar{x} \qquad (3)$$

In a similar manner, the component parallel to the Y axis is found to be

$$P_y = \omega^2 m\bar{y} \qquad (4)$$

The resultant force is

$$P = m\bar{r}\omega^2 = \frac{mv^2}{\bar{r}} \qquad \text{Formula XLI}$$

If the mass m in Formula XLI is given in slugs (32.174 pounds), the radius r in feet, and the velocity in feet per second, the resultant force is in pounds.

The formula may be written

$$P = \frac{W}{32.174}\bar{r}\omega^2 = \frac{W}{32.174} \times \frac{v^2}{r}$$

in which W is the mass in pounds, P is force in pounds, and the other units are feet and seconds.

(Formula XLI applies to the cgs system and to the fps system to give dynes and poundals, respectively.)

When any body is rotating about an axis with an angular velocity ω, a centripetal resolution equation can always be set up by equating the sum of all force components acting on the body in the direction from the center of mass to the center of rotation to the product $W\bar{r}\omega^2/g$ or to $Wv^2/g\bar{r}$, v being the tangential velocity of the mass center. This is the meaning of Formula XLI. This procedure is a correct one in all cases, regardless of the complexity of the force system, the direction of rotation, and whether the body has or has not an angular acceleration. This result is simply a special case of the proposition of Art. 145, which states that the product of the entire mass multiplied by the acceleration of its center of gravity is equal to the sum of the products of the elements of mass of the body multiplied each by its acceleration. It has been separately derived here because of its extreme importance in applied mechanics.

Example 1

If the body of Fig. 300 at any instant has an angular velocity ω, with O as the center,

$$Q - P\cos\phi = \frac{W\bar{r}\omega^2}{g}$$

Example 2

A light stiff rod attached to a horizontal shaft rotating with uniform velocity extends perpendicularly from the shaft and carries a mass of 64.35 lb with its center of mass 5 ft from the center of the shaft. The velocity of the center of gravity of the mass is 20 ft per sec. Find the tension in the rod when it is horizontal, when it is vertical above the shaft, and when it is at an angle of 30° above the horizontal.

The resultant of the tension in the rod and the force of gravity is toward the center of the shaft and is

$$\frac{Wv^2}{\bar{r}g} = \frac{64.35}{32.174} \times \frac{20^2}{5} = \frac{2 \times 400}{5} = 160 \text{ lb}$$

When the rod is horizontal the tension is 160 lb. When it is vertical above the axis, the effective force downward is 160 lb, of which 64.35 lb is from the weight of the body. If the tension is represented by Q,

$$Q + 64.35 = 160$$
$$Q = 95.65 \text{ lb}$$

When the rod is vertical downward the weight increases the tension.

$$Q = 64.35 + 160 = 224.35 \text{ lb}$$

At 30° above the horizontal,

$$Q + 64.35 \sin 30° = 160$$
$$Q = 127.83 \text{ lb}$$

Problems

1. The mass of the example makes 1 rev per sec about a vertical **axis.** Find the tension in the rod at each position. *Ans.* $P = 2 \times 5 \times 4 \times 9.87 = 394.8$ lb.
2. A mass of 20 lb is whirled in a vertical plane at the end of **a rod** 4 ft long with a uniform speed of 16 ft per sec. Find the tension at the top and at the bottom.
3. The mass and rod of Prob. 2 are stationary vertically above the axis and are allowed to fall, turning freely about a hinge at the end of the rod. Considering the combined mass concentrated at the center of gravity, find the tension at the bottom of the path.

$$\textit{Ans.}\ P = \frac{20}{g} \times \frac{v^2}{4} \qquad \frac{20v^2}{2g} = 20 \times 8 \qquad \frac{20v^2}{g} = 320$$
$$P = \frac{320}{4} = 80 \text{ lb} \qquad Q = 100.$$

4. Solve Prob. 3 if the mass has a velocity of 12 ft per sec. at the top.

$$\textit{Ans.}\ \frac{Wv^2}{2g} = W\left(\frac{144}{2g} + 8\right) \qquad \frac{Wv^2}{g \times 4} = \frac{20}{4}\left(\frac{144}{g} + 16\right) = 102.38 \text{ lb}$$
$$Q = 122.38 \text{ lb.}$$

5. Solve Prob. 3 for the tension when the rod is horizontal and when it is 30° below the horizontal. *Ans.* $Q = 40$ lb; $Q = 70$ lb.
6. Show that the tension in Prob. 3 is independent of the length of the rod.

When a vehicle rolling on the ground travels in a curved path there are, neglecting windage, only the force of gravity and the forces exerted by the ground acting on it. The ground forces can be resolved into normal forces perpendicular to the ground surface and friction forces parallel to the surface. The force diagram for Prob. 7 below should show the weight of the boy and the bicycle acting downward at the center of gravity, the normal forces of the ground upward under the wheels, and the friction forces at the bottoms of the wheels acting toward the center of curvature. For a practical solution of the problem, the entire mass may be assumed at the center of gravity. Under this assumption, the moment of inertia about the center of gravity is zero and, by Formula XL, the moment of all forces acting must have zero moment with respect to this point. Total normal force under the wheels is equal to the weight of the boy and the bicycle because there is no upward acceleration. (For this problem the bicycle may be considered the same as a unicycle.) The friction force under the wheel is the only force toward the center and must account for the acceleration toward the center. The equation of moments about the center gives the angle required. Do not resort to the use of the fictitious "centrifugal force."

Problems

7. A boy on a bicycle makes a turn on a level street with a speed of 20 ft per sec. If the radius measured to the center of gravity of the boy and the bicycle is 40 ft, how much must he lean from the vertical and what must be the minimum coefficient of friction between the wheels and the ground?

Ans. 17°16′; 0.31.*

8. How fast can the boy of Prob. 7 increase his speed if the normal force under the rear wheel is 0.6 of the weight and the coefficient of friction under that wheel is 0.5? (Assume that the centripetal force is still evenly divided between the wheels.)

Ans. 8 ft per sec per sec, and less as his speed increases.

9. A truck makes a turn of 200-ft radius on a level road. The normal force under the inner wheels is zero. The total normal force equals the weight. Moments of all forces about the center of gravity equal zero. Making no allowance for the yielding of tires and springs, find the speed at which the truck will upset and find the coefficient of friction between the road and the tires required to justify the assumption of upsetting.

10. If the distance between the rails of a track is 56.5 in., how much must the outer rail be elevated for a curve of 800-ft radius in order that the resultant force may be normal at a speed of 30 mph? *Ans.* 4.25 in.

11. A dynamo armature weighing 400 lb runs at 1200 rpm. If its center of gravity is 0.03 in. from its axis, what is the maximum vertical pull upward on the bearings, and what is the maximum horizontal force on the bearings?

Ans. 90.8 lb; 490.8 lb.

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* This solution neglects (*a*) the fact that the mass is not actually concentrated at a point, (*b*) a minute effect on the angle due to the gyroscopic effect of the rotating wheels, sprocket, etc., and (*c*) the still more minute effect of the rotation of the earth. This third effect would be greatest at the pole, and the student can easily find what it would be there by considering that the angular velocity would be 2π rad per day more or less than the 0.5 rad per sec of the problem as given.

12. A small 4-lb ball is hung from a point by a string so that the distance from the point to the center of the ball is 5 ft. If the ball is set moving in a horizontal circle of 6-ft diameter, what is its velocity and what is the tension in the string?

Ans. $\frac{3}{2}\sqrt{g}$; 5 lb.

(The vertical component of the tension balances the weight. The horizontal component is the centripetal force.)

13. If it is necessary to move the top end of the string around in a circle of 1-in. radius in order to maintain steady motion of the ball of Prob. 12, what is the wind resistance on the ball?

Ans. $\frac{1}{12}$ lb.

160. Kinetic Energy of Rotation. Figure 304 shows an element of mass dm at a distance r from an axis through O. If this element rotates about the axis with an angular velocity ω, its linear velocity is $r\omega$ and its kinetic energy is

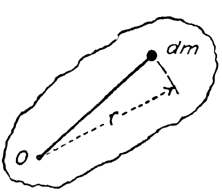

FIG. 304.

$$dU = \frac{r^2\omega^2\,dm}{2} \tag{1}$$

The kinetic energy of the entire body rotating with angular velocity ω about the axis through O is the integral of Eq. (1).

$$U = \frac{\omega^2}{2}\int r^2\,dm = \frac{I\omega^2}{2} \qquad \text{Formula XLII}$$

If the kinetic energy is in foot-pounds, $I = mk^2$, in which m is the mass in geepounds (slugs) and k is the radius of gyration in feet. The formula may be written

$$U = \frac{W}{32.174} \times \frac{k^2\omega^2}{2} \qquad \text{Formula XLIII}$$

in which W is the mass in pounds and U is the energy in foot-pounds.

Problems

1. A wheel 4 ft in diameter weighs 321.7 lb and has a radius of gyration of 1.8 ft. What is the kinetic energy in foot-pounds when the rim of the wheel is moving with a speed of 48 ft per sec?

Ans. $U = \frac{10 \times 3.24 \times 576}{2} = 9331.2$ ft-lb.

2. The wheel of Prob. 1 is given a rim velocity of 48 ft per sec in 8 sec by means of a small cord wound around it. What is the tension in the cord? What is the distance traveled from rest? What is the total work done by the cord?

Ans. $P = 48.6$ lb; $s = 192$ ft; $U = 9331.2$ ft-lb.

3. A homogeneous solid cylinder 4 ft. in diameter and weighing 600 lb revolves on its axis at the rate of 2 rev per sec. Find the kinetic energy in foot-pounds.

Ans. $U = \frac{600 \times 2 \times 16 \times 9.8696}{32.174 \times 2} = 2945$ ft-lb.

4. A homogeneous hollow cylinder has an outside diameter of 4 ft and an inside diameter of 3 ft. It weighs 400 lb and rotates on its axis at the rate of 300 rpm. Find its kinetic energy in foot-pounds. *Ans.* $U = 19{,}172$ ft-lb.
5. Find the kinetic energy of the pulley of Prob. 2 of Art. 157. Find the distance required to stop the pulley by means of the work done. Compare with the answer.
6. A homogeneous solid cylinder, which is 4 ft in diameter and weighs 400 lb, rotates on frictionless bearings. A rope wound round the cylinder supports a mass of 120 lb on the free end (Fig. 282). Find the angular velocity of the cylinder and the linear velocity of the rope after the mass has moved 60 ft from rest.

$$\textit{Ans.}\ \frac{800\omega^2}{2g} + \frac{120v^2}{2g} = 7200$$

$$\frac{1280\omega^2}{2g} = 7200$$

$$\omega = 19.0 \text{ rad per sec};\ v = 38.0 \text{ ft per sec.}$$

7 A uniform rod AB is 6 ft in length and is hinged at A. The rod is placed in the position of unstable equilibrium and then released. Find its velocity when it reaches the horizontal position of Fig. 305.

Ans. Velocity of B is 24.06 ft per sec.

8 Solve Prob. 7 for the velocity when the rod is 30° below the horizontal and when the rod is vertical downward.

Ans. 4.91 rad per sec; 5.67 rad per sec.

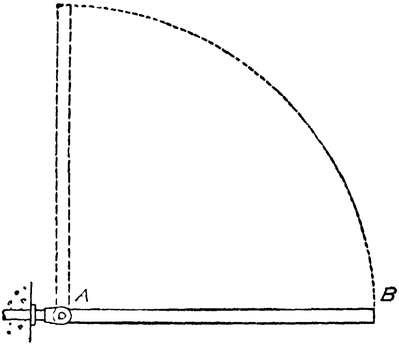

FIG. 305.

161. Rotation and Translation.

In the derivation of Formula XLII of Art. 160, O in Fig. 304 was the center of rotation of the body and had no velocity (relative to the frame of reference for the body). The point O was not necessarily at the center of gravity of the body but could just as well, so far as the derivation is concerned, be a fixed hinge exerting forces on the body or an instantaneous center such as point B of Fig. 302. If there is no slipping at B between the wheel and the plane of Fig. 302, the kinetic energy of the wheel is expressed by Formula XLII by substituting the angular velocity of the wheel and its moment of inertia (in geepounds feet^2) with respect to the point B in the formula. If Formula XLIII is to be used, the radius of gyration must be with respect to point B.

When it is more convenient, the moment of inertia of a rolling wheel can be expressed by adding the "kinetic energy of translation," $mv^2/2g$, with m the mass in pounds and v the velocity *of the center* in feet per second, to the "kinetic energy or rotation," $I\omega^2/2g$, with I in pounds feet^2 with respect to an *axis through the mass center* parallel to the axis of rotation at the instantaneous center. This proposition is called *Koenig's principle.* In the simple derivation which follows, I is in geepounds feet squared, and m is in geepounds.

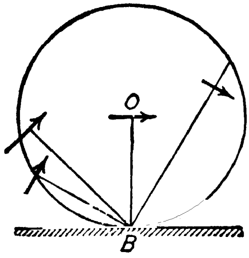

Fig. 283. (*Repeated.*)

Figure 283 shows a cylindrical body of radius r, which is rolling with velocity v on a plane surface. The angular velocity of the body is ω and the instantaneous linear velocity of any element of mass of the body is $r'\omega$, where r' is the distance of the element from the instantaneous center B. Formula XLII gives the result of summing up the kinetic energies of all the elements of mass

$$U = \frac{I_b\omega^2}{2} \tag{1}$$

$$I_b = I_c + mr^2 \tag{2}$$

Substituting this value of I_b in Eq. (1),

$$U = \frac{(I_c + mr^2)\omega^2}{2} \tag{3}$$

The linear velocity of the center is $r\omega = v$, and when this substitution is made Eq. (4) becomes

$$U = \frac{I_c\omega^2}{2} + \frac{mv^2}{2} \tag{4}$$

The derivation was simplified by using a cylindrical body rolling on a plane surface, but the proposition is true for more general cases and may be used for a body of any shape if OB is constant.

The student should keep Eq. (4) in mind as an idea rather than as a formula. If he thinks of it as a formula he should visualize the denominators as $2g$ so as to avoid the danger of omitting the g in computing I and m. If I and m are computed in pounds mass units, the energy will be in foot poundals when g is omitted from the formulas.

If r is the radius from the axis through the center of mass of the cylinder to the line of contact with the plane surface, $v = r\omega$, and Eq. (4) may be written

$$U = \frac{I_c\omega^2}{2} + \frac{mr^2\omega^2}{2} \tag{5}$$

$$U = \frac{m\omega^2}{2}(k_c^2 + r^2) = \frac{m\omega^2k^2}{2} = \frac{I\omega^2}{2} \tag{6}$$

in which k_c is the radius of gyration with respect to the axis through the center of gravity, and k is the radius of gyration with respect to the line of contact with the surface. These equations show that the energy computed from the linear velocity of the center of mass and the angular velocity about that center is the same as the energy computed from the angular velocity about the line of contact. When there is no line of contact or when slippage is taking place at the line of contact, the true instantaneous center must be used for the axis of moment of inertia in Formulas XLII and XLIII.

Problems

1. A homogeneous solid cylinder is 4 ft in diameter and weighs 240 lb. It is rolling at a speed of 60 ft per sec. Find the kinetic energy of translation, the kinetic energy of rotation, and the total kinetic energy. *Ans.* 13,427 ft-lb; 6713 ft-lb; 20,140 ft-lb.
2. When a homogeneous cylinder is rolling without slipping on a surface, show that the kinetic energy of rotation is one-half the kinetic energy of translation.
3. A homogeneous solid cylinder rolls down an inclined plane which makes an angle of 20° with the horizontal. What velocity will it acquire while moving 100 ft from rest? What velocity would it acquire if it were to slide down the plane with no friction? *Ans.* 38.3 ft per sec; 46.9 ft per sec.
4. In Prob. 3, what is the average velocity when the cylinder rolls down the plane? What is the time of descent? *Ans.* $t = 5.22$ sec.
5. The outside diameter of a hollow cylinder is 4 ft and the inside diameter is 3.9 ft. The cylinder rolls 100 ft down a plane which makes an angle of 10° with the horizontal. Find the final velocity. *Ans.* 23.78 ft-per sec.
6. A homogeneous solid cylinder, which is 4 ft in diameter and weighs 600 lb, rolls on a plane surface with a velocity of 20 ft per sec. Find the kinetic energy in foot-pounds, regarding the axis as moving 20 ft per sec and the cylinder as rotating on its axis at the rate of 10 rad per sec. Also find the kinetic energy, regarding the cylinder as rotating about the axis of contact with the surface with an angular velocity of 10 rad per sec. *Ans.* $U = 5594.6$ ft-lb.
7. A bar of length l is rotating with angular velocity ω about an axis through one end perpendicular to its length. Find its kinetic energy.
8. Solve Prob. 7 from the condition that the center of gravity of the bar has a linear velocity of $l\omega/2$ and the bar rotates about the center with an angular velocity of ω rad per sec. *Ans.* $U = Wl^2\omega^2/6g$.
9. A car weighing 200 lb has a radius of gyration of 2 ft with respect to the vertical axis through its center of gravity. The car moves with a velocity of 40 ft per sec on a straight track. If no energy is lost in friction, what is its velocity when it runs onto a curve of 10-ft radius? *Ans.* 39.2 ft per sec.

162. Dynamic Balance. When the elements of mass constituting a rotating device supported by bearings are so arranged that the reactions at the bearings are the same when the device is rotating as they are when it is stationary, the device is said to be in *dynamic balance* or *running balance*. When such a device is not in dynamic balance, the bearing reactions are called the *static reactions* when the device is not rotating and the additional forces which the bearings must exert when the device is rotating are called the *dynamic reactions*. For dynamic balance the centrifugal forces exerted by the elements of mass on the axis of rotation must constitute a system in static equilibrium. The rotor of Prob. 11, Art. 159, although only 0.03 inches off center, is badly out of balance and produces, at the speed given, a dynamic force of nearly 500 pounds. If its speed were increased to 6000 revolutions per minute, the dynamic reactions would be over 6 tons.

Figure 306,I represents a rotating shaft with one type of unbalance. When the shaft is rotated at a velocity of ω radians per second, the centrifugal force exerted on the shaft by the mass W' is $P' = W'b\omega^2/g$, and that exerted by W is $P = Wa\omega^2/g$. The system of centrifugal forces, P' and P, cannot possibly be in equilibrium since the two forces are opposite and not along the same line.

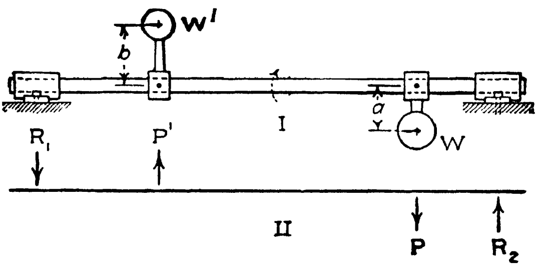

Fig. 306.

If the two eccentric masses of Fig. 306,I are so proportioned that $W'b = Wa$, the forces P' and P of Fig. 306,II are equal and constitute a couple. The shaft is then in static balance and will not turn from any position under the influence of gravity even though the bearings are entirely frictionless, but dynamically it is in a state of "couple unbalance" and requires at least two "counterweights" for running balance. If the two products are not equal, the forces P' and P of Fig. 306,II are unequal and have a resultant which lies somewhere to the right of P or to the left of P'. This resultant may easily fall outside the bearings so that it will not be practical to produce running balance by a single counterweight. In either case, dynamic balance can be produced by two counterweights, one in each of two planes, selected at convenience, perpendicular to the shaft.

Suppose the shaft of Fig. 306 is to be put in running balance by a mass M at a distance n from the center of the shaft in a plane near the mass W and a second mass M' at a distance n' from the center of the shaft and in a plane between the left bearing and W'. (To avoid negative values, let n be upward as seen in the figure and n' downward.) Let b' be the distance from the left balance plane to W' and a' the distance from the left balance plane to W. Let c be the distance between the two balance planes. Let Q be the centrifugal force, at any shaft speed, of M and Q' be the centrifugal force of M'. Assume that the system of centrifugal forces exerted by the four masses is in equilibrium, that is, that the dynamic forces of the bearings are zero. Taking moments about an axis in the left balance plane perpendicular to the plane containing the four mass centers,

$$P'b' + Qc = Pa' \tag{1}$$

Expressing the forces in terms of their factors,

$$\frac{W'bb'\omega^2}{g} + \frac{Mnc\omega^2}{g} = \frac{Waa'\omega^2}{g} \tag{2}$$

From an inspection of Eq. (2) it is obvious that g and ω^2 can be canceled out and might as well have been left out. The significance of canceling out g and ω^2 is that it is immaterial what units are used so long as the same units are used for all like quantities, and that if balance is effected for one speed the shaft is in balance at all speeds.

Solving Eq. (2) for Mn,

$$Mn = \frac{Waa' - W'bb'}{c} \tag{3}$$

The product Mn is a vector quantity called, usually, the *counterbalance*. M or n may be chosen at will and the other quantity selected to give the proper product. Wa and $W'b$ are vector quantities like Mn and are called *unbalances*. Instead of adding the counterbalance Mn in the chosen plane, mass may be removed from the opposite side of the rotor so that the product of the mass removed times its distance from the axis equals Mn. This leads naturally to a concept of an unbalance *in this plane*. It has become customary to speak of an unbalance in a given plane, but the expression is ambiguous unless it is stated or understood where the cognate plane is located or unless it is specifically stated that the plane designated is the unique plane in which a single counterbalance will effect dynamic balance for the whole rotating mass.

When one of the required counterbalances has been determined, the other may be determined by using any other axis or by simply making the other counterbalance of the proper amount to produce static balance. It is well to determine the second counterbalance by moments of the centrifugal forces and to check by gravity moments about the axis of the rotor. When dynamic balance has been secured, the rotor is always in static balance but the converse statement is true only when the rotor is a thin disk.

The statement made above as to the significance of canceling is strictly correct only for a perfectly rigid body. No body is perfectly rigid, and when a set of counterbalances has been added to a rotor, the centrifugal forces bend the rotor so that the distance terms of the unbalances and counterbalances are changed. At high speeds, small changes in these distances cause large changes in the forces. A shaft or rotor of low rigidity balanced for a particular speed may be noticeably out of balance at other speeds, unless the counterbalances are exactly opposite to and symmetrical with the unbalances, an arrangement which is not usually practical.

The outline above was based on a pair of unbalances lying in a plane containing the axis of the shaft. If, in Fig, 306, the mass W is not at 180° (or 0°) around the shaft from mass W', the counterbalance as stated in Eq. (3) is not correct. In such a situation two sets of balancing equations have to be written and solved. Each unbalance is resolved into two components, one in each of a pair of perpendicular planes through the axis of the shaft. Each of these planes is balanced in the manner indicated above to find two counterbalances at right angles to each other in each of the selected planes perpendicular to the shaft. Finally, the two counterbalances in each of these balance planes are added vectorially to give the single counterbalance, magnitude and direction, needed in that plane.

When a body designed to rotate has large off-center masses of known size and location such as the cheeks and crankpins of a crankshaft, it is not difficult to add to the design a set of suitable counterbalances, but since there are always errors in manufacture, the final product is not likely to be in good dynamic balance unless subjected to a final balancing operation. Since the residual unbalance is never known, this final process consists of mounting the body in suitable "floating" bearings, spinning it, and measuring in some way the forces or motions resulting at the bearings or elsewhere. There are a number of machines of different designs for this purpose, some of them so nearly automatic that an operator of no particular skill and no knowledge of the principles involved can balance rotors at a cost of a few cents each. On the other hand there are difficult cases, especially where there are more than two bearings along with heating effects, which baffle the most expert engineers. "Perfect balance," like any other perfect thing, is an ideal never fully attained.

Example 1

In Fig. 306, suppose $W' = 3$ lb, $b = 4$ in., $W = 2$ lb, and $a = 3$ in., and suppose it is 20 in. along the shaft from one mass to the other. Find the counterweights necessary for dynamic balance if they are to be in planes perpendicular to the shaft 4 in. outside the unbalances. Omitting g and ω^2, and taking masses in pounds and distances in inches, $W'b = 12$ and $Wa = 6$. Let X be the left counterbalance, assumed below the shaft, and Y be the right counterbalance, assumed above the shaft.

$$28Y + 4 \times 12 = 24 \times 6 \qquad Y = 3\tfrac{3}{7} \text{ lb-in.}$$
$$28X + 4 \times 6 = 24 \times 12 \qquad X = 9\tfrac{3}{7} \text{ lb-in.}$$
$$X + 6 = Y + 12. \quad \text{Static balance.} \quad \text{Check.}$$

The right counterweight is $3\tfrac{3}{7}/6 = \tfrac{4}{7}$ lb, and the left counterweight is

$$\frac{9\tfrac{3}{7}}{6} = 1\tfrac{4}{7} \text{ lb}$$

Example 2

In Prob. 11 of Art. 159, the shaft is 60 in. long from center to center of the bearings. Suppose the center of mass of every section of the rotor to be the same distance from the axis of the shaft as that of every other section and the center of mass of the whole armature to be 32 in. from the left bearing. With this disposition, the whole mass is equivalent to a point mass 32 in. from the left bearing and 0.03 in. from the axis. Assume that static balance has been secured by adding 2.4 lb of lead inside a collar located 10 in. from the right bearing. The center of gravity of the lead is 5 in. from the axis. Find the reactions of the bearings when the armature is making 1200 rpm.

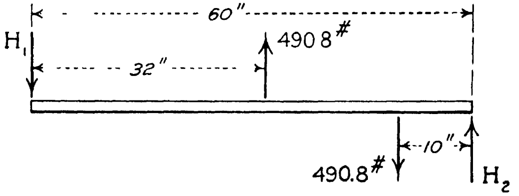

FIG. 307.

The centrifugal force of the mass of the armature is 490.8 lb. The centrifugal force of the 2.4 lb of lead is also 490.8 lb. In Fig. 307, these centrifugal forces are horizontal. The horizontal reactions of the bearings are H_1 and H_2. By moments about the vertical lines through the bearings, H_1 is found to be 147.2 lb in one direction and H_2 is found to be an equal force in the opposite direction. When the center of gravity is below the axis, the centrifugal force adds 147.2 lb to the load on the left bearing and subtracts the same amount from the load on the right bearing.

Problems

1. Find the reactions of the bearings of the above example when the center of gravity of the armature is directly below the axis. *Ans.* 334.3 lb; 68.1 lb.

2. Solve Prob. 1 when the center of gravity of the armature is directly above the axis of the shaft.

3. Assume the 2.4-lb counterweight removed from the rotor of Example 2 and find the counterweights required 5 in. from the axis in planes 10 in. from the right bearing and 8 in. from the left bearing.

4. In Example 1 assume $W'b$ to remain in the plane of the paper and Wa to be turned 90° on the shaft toward the observer. With other data as in Example 1, find the proper counterbalances.

5. The framework of Fig. 308 is to be rotated about the vertical axis. Find the lengths x and y for dynamic balance, assuming all bars to weigh 2 lb per foot of length. In taking moments, integrate the moments of the centrifugal forces of the elements of the inclined bar. The total centrifugal force of this bar is the same as though the mass were concentrated at its midpoint, but the resultant centrifugal force has a moment arm of 2 ft with respect to an axis at the top of the bar. Finally check the problem by determining that the frame is in static balance.

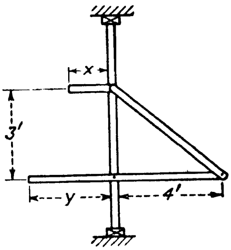

FIG. 308.

163. Center of Percussion. A body free to turn about a hinge (either a fixed center or an instantaneous center) has a point beyond the mass center on the line connecting the mass center and the hinge called the *center of percussion.* A force through this point produces moments about the hinge and about the mass center proportional to the moments of inertia of the body with respect to these two points. If the force is normal to the line through the hinge and the mass center it produces no additional force at the hinge. Traditionally, in speaking of this property, the hinge has been called *the center of suspension.* The center of percussion is not a fixed point in the body but is at a position depending on the center of suspension and moves off to infinity as the center of suspension approaches the center of mass. Since a force applied at the center of percussion produces no force at the center of suspension, there need be no actual restraint at that point to keep it in position when an impulse is applied at the center of percussion. A body placed on a smooth plane and struck by a blow parallel to the plane begins its motion by turning about an instantaneous center corresponding to a center of percussion on the line of action of the blow.

In addition to the interesting practical aspects of the center of percussion such as the desirability of holding a baseball bat, club, or any striking tool at the center of suspension corresponding to the center of percussion which is to deliver the blow, or the suitability of the center of percussion as the place to locate the wheels of a two-wheeled vehicle, the point has some important applications in analysis. Some of these will be noted in connection with moving reference frames for problems of dynamics. Of more immediate importance is the proposition: *The resultant of all the forces acting on a rigid body passes through the center of percussion corresponding to the center of rotation of the body.* The student may use this proposition as a check on his solutions.

To locate the center of percussion, suppose a body such as the bar of Fig. 309 is suspended from a hinge and subjected to a sudden horizontal force. Let P be the force applied to the body, a the acceleration of the center of mass, and N the component of the hinge reaction opposite the direction of P. Then Formula XXXI gives

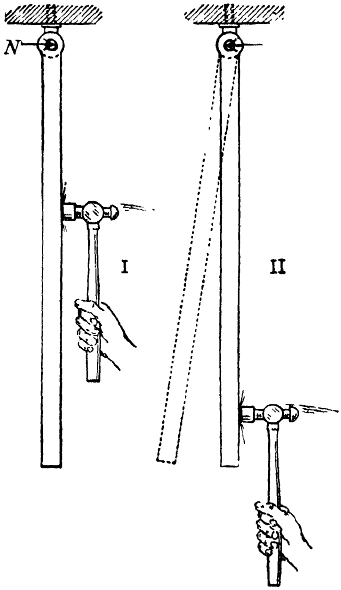

FIG. 309.

$$P - N = ma \tag{1}$$

If $N = 0$,

$$P = ma \tag{2}$$

If the force P acts at a distance l from the axis, its torque is Pl. By Formula XXXII,

$$Pl = I\alpha \tag{3}$$

When the angular acceleration is α, the linear acceleration of the center of mass is $\bar{r}\alpha$. When $P = m\bar{r}\alpha$ is substituted in Eq. (3), the result is

$$m\bar{r}\alpha l = I\alpha \tag{4}$$

$$m\bar{r}l = I = mk^2 \tag{5}$$

$$l = \frac{k^2}{\bar{r}} \tag{6}$$

Equation (6) states that the center of percussion is located at a distance from the axis equal to the square of the radius of gyration with respect to the axis divided by the distance from the axis to the center of gravity.

Problems

1. A uniform rod of length L is supported by an axis through one end perpendicular to its length. Find the center of percussion.

   Ans. $l = \frac{L^2/3}{L/2} = \frac{2L}{3}$ from the support.

2. Find the center of percussion of a uniform rod suspended by an axis at one-third the length from the end.

   Ans. Center of percussion is at end of rod $2L/3$ from support.

3. Find the center of percussion of a uniform rod suspended on an axis at one-fourth of the length from the end.

   Ans. $l = 7L/12$; center of percussion is $L/6$ from opposite end.

4. Derive the expression for the location of the center of percussion by the formula $T = I\alpha$, using Eq. (3) and also moment about the center $= I_c\alpha$.
5. Locate the center of percussion of a homogeneous disk of radius a for an axis of suspension at the periphery. *Ans.* $\frac{3}{2}a$.
6. Find the center of percussion of a disk with the axis of suspension off the disk at a distance b from the center.
7. Using Fig. 300 and the value for N of Eq. (5) of Art. 158, show that the resultant of all forces passes through the center of percussion.

164. Application to Miscellaneous Problems. Many problems of dynamics, as applied to engineering, comprise a set of initial conditions from which are to be deduced or predicted another set of conditions after a lapse of time during which some sort of action takes place. Such problems nearly always require an *assumption* as to what will take place and an *analysis* of the validity of that assumption. Sometimes the action to be expected is so obvious that only one assumption can reasonably be made, but in other cases more than one possibility exists and each must be tried until a solution has been found which is wholly consistent. The several phases of action often have to be considered as a succession of related problems.

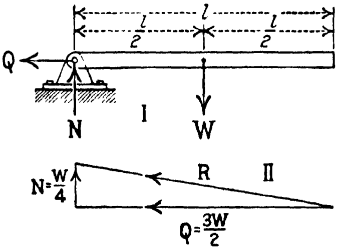

FIG. 310.

Example 1

A uniform slender bar of length l is hinged at one end. The bar is placed vertically above the hinge and turns from the vertical position without friction. Find the hinge reaction when it has reached the horizontal position as shown in Fig. 310,I.

The component Q depends upon the velocity, and this can best be found by an energy equation

$$\frac{Wl}{2} = \frac{I\omega^2}{2}$$

$$Wl = I\omega^2 = \frac{W}{g} \times \frac{l^2}{3}\omega^2$$

$$\frac{\omega^2}{g} = \frac{3}{l}$$

$$Q = mr\omega^2 = \frac{W}{g} \times \frac{l}{2}\omega^2$$

$$Q = \frac{Wl}{2} \times \frac{\omega^2}{g} = \frac{Wl}{2} \times \frac{3}{l} = \frac{3W}{2}$$

The component N is independent of the velocity but depends on the acceleration. The angular acceleration can be found by taking moments about the pin.

$$T = \frac{Wl}{2} \qquad I = \frac{Wl^2}{3g}$$

$$\alpha = \frac{3g}{2l}$$

$$a = \bar{r}\alpha = \frac{l}{2} \times \frac{3g}{2l} = \frac{3g}{4}$$

$$N = W - \frac{Wa}{g} = W - \frac{W}{g} \times \frac{3g}{4} = \frac{W}{4}$$

Problems

1. The bar of Fig. 310 starts from rest vertically above the hinge and falls 60° to the position shown in Fig. 311. Find the components of the hinge reaction.

Ans. $Q = W/4$; $N = 0.2163W$.

2. Solve Prob. 1 when the bar is 30° below the horizontal.

3. Solve Prob. 1 when the bar is hanging vertically downward.

Ans. $Q = 4W$; $N = 0$.

4. Find the hinge reactions for Example 2 of Art. 157.

Ans. Reaction = 280 lb vertically upward.

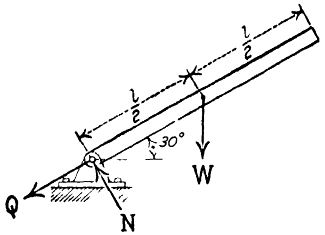

FIG. 311.

5. A homogeneous solid cylinder of radius r rolls down a 20° inclined plane. The friction is sufficient to prevent sliding. Find its acceleration in radians per second per second and the acceleration of its axis in feet per second per second. Solve by considering the cylinder to be rotating about its line of contact with the plane.

$$\textit{Ans.}\ \alpha = \frac{2g \sin 20°}{3r} = \frac{7.34}{r} \text{ rad per sec per sec}$$

$$a = 7.34 \text{ ft per sec per sec}$$

6. A truck with a wheel base of 10 ft and the center of gravity midway between the wheels and 4 ft from the road is proceeding down a slope of 5 vertically on

12 horizontally when the rear-wheel brakes are set to skid the rear wheels. If the truck is going at 20 ft per sec and the coefficient of friction is 0.6, how far does it go in the next 4 sec? *Ans.* 134 ft.

7. A motorcyclist with all the power he can use is climbing a hill of slope 5 vertically on 12 horizontally. If the coefficient of friction under the driving (rear) wheel is 0.8, the wheel base is 5 ft and the center of gravity of machine, and the rider is 2 ft above the road and 1.5 ft forward from the rear axle, how fast can he accelerate? *Ans.* $4g/13$.

8. A hand truck with wheels in front and legs at the rear 4 ft away has a load center 1.5 ft back of the wheels and 1 ft above the floor. It is given a forward velocity of 8 ft per sec and comes to rest by friction between the legs and the floor. Assume the coefficient as 0.25 and neglect the mass and rolling resistance of the wheels. Find the distance required to stop.
Ans. 10.67 ft (if g is 32).

9. A 24-lb gasoline-powered toy automobile is constrained to run in a circle of 12-ft radius by means of a cord fastened to the top of a stake 5 ft high. The car drives on all four wheels and the coefficient of friction between wheels and road is 0.4. When it attains a speed of 24 ft per sec it can go no faster because the wheels are slipping. Find the wind resistance. (Assume g as 32 ft per sec per sec.) *Ans.* 3.6 lb.

10. A wheel and axle, together weighing 192 lb, were placed on a pair of 30° inclined rails and rolled 18 ft in 36 sec, rolling on the axle which was 3 in. in diameter. Neglect rolling resistance and find the moment of inertia of the body.

11. Show that when a small circle rolls on a larger one, either internally or externally, the velocity of the center of the smaller circle in its path is $r\omega$, where r is the radius of the small circle and ω its angular velocity. (Divide the total length of the path by the number of revolutions of the small circle per rotation.)

12. Show that when a solid homogeneous sphere rolls without slipping on a plane surface the kinetic energy of rotation is 0.4 of the kinetic energy of translation.

13. Make use of the results of Probs. 11 and 12 to find the height above the bottom of the loop from which a ball 1 in. in diameter will roll down and around a 6-in. diameter "loop the loop" with sufficient velocity to maintain a pressure equal to one-half the weight of the ball against the race as it passes over the top. (Work with pounds, inches, and seconds.)
Ans. 8.125 in. to center of ball.

14. If the ball of Prob. 13 weighed 2 lb, what were the normal and friction forces between it and the race when the center of the ball was moving vertically upward? *Ans.* $N = 5.875$ lb; $F = 0.556$ lb.

15. A hollow inverted cone 3 ft high and 8 ft in diameter is arranged to rotate about its axis (vertical). If the coefficient of friction between the cone and a small body inside halfway up the slope is 0.25, what are the maximum and minimum speeds of rotation possible without dislodging the body?
Ans. $\omega = \sqrt{8g/13}$, $\sqrt{4g/19}$.

16. A racing car goes around a turn banked at 30° with the horizontal. If the radius of curvature is 300 ft and the coefficient of friction is 0.4, find the skidding speed.

17. A cone 5 in. high and 12 in. in diameter is rotated about its vertical axis at 4 rad per sec. What must be the coefficient of friction to enable a small body to ride around on the cone halfway down the slope? (Resolve in directions parallel and normal to the slope using the components of the centripetal acceleration in these directions. Avoid fictitious forces.)

18. A small body is riding around 1.5 ft from the center of a horizontal turntable with an angular velocity of 2 rad per sec. If the coefficient of friction is $\frac{5}{16}$, find (*a*) the maximum angular acceleration of the table possible without dislodging the body, and (*b*) the maximum vertical linear acceleration of the table downward without dislodging. (Let g = 32 ft per sec per sec.)

Ans. (*a*) 5.33 rad per sec per sec; (*b*) 12.8 ft per sec per sec.

19. A 40-lb disk 4 ft in diameter is attached to a light bar in the plane of the disk and extending to a hinge 5 ft from the center of the disk. Draw a sketch with the center of the disk 4 ft to the left of and 3 ft above the hinge. The device is being turned by a cord attached to the bar, passing around the disk, and coming off tangentially to the right at the top. If the tension in the cord is 140 lb and the velocity of rotation is 4 rad per sec, what are the components of the hinge reaction?

Ans. Parallel to the bar, 36.6 lb; normal to the bar, 48 lb.

20. The light rod of the device of Prob. 19 is replaced by a slender rod weighing 40 lb. The disk is placed directly above the hinge and a cord is attached to the mid-point of the rod (1.5 ft above the hinge). This cord extends horizontally to the right, over a light smooth pulley and down to a suspended mass of 60 lb. If the disk is released what will be the tension in the cord and the reaction at the hinge an instant later, assuming the disk has not moved appreciably and has not acquired an appreciable velocity? (Because of the right angle between the cord and the rod, the acceleration of the suspended mass is the same as the tangential acceleration of the mid-point of the rod. In making the horizontal resolution, the sum of the horizontal forces can be equated to the sum of the two masses, bar and disk, each multiplied by the acceleration of its center.)

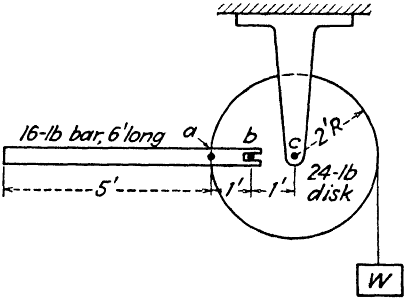

FIG. 312.

21. Figure 312 shows a slender 16-lb bar 6 ft long attached by two pins to a 24-lb disk 4 ft in diameter. Pin *a* is attached to the disk at its periphery and passes through a smooth hole in the rod. Pin *b* is rigidly attached to the disk and fits in a smooth rectangular notch at the end of the bar. Neglect all friction. Find *W* to give the disk and bar an angular acceleration of 6 rad per sec per sec when the bar is in the position shown. If the bar is pulled down to the vertical position and released, what will be the force components at *a*, *b*, and *c* as it passes the horizontal position shown?

22. Solve Prob. 21 if the cord is not attached to the disk but passes over the disk and is attached to the mid-point of the bar. Neglect friction between the cord and the disk.

23. The device in Fig. 313 is released from the position shown. Find the reaction at the hinge when the bar arrives at the vertical position. *Ans.* 6.4 lb to the right.

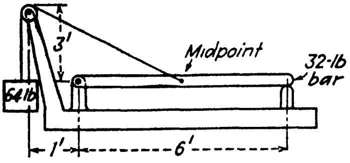

FIG. 313.

24. A wheel weighing 12 lb has a radius of 1 ft and a radius of gyration of 0.707 ft. Its axle, of radius 0.13 ft, is placed in bearings with a coefficient of friction of $\frac{5}{12}$. A cord passed over the wheel has suspended masses 8 lb on one side and 12 lb on the other. Find the angular acceleration, assuming the cord does not slip on the wheel. *Ans.* $4g/43$.

25. A homogeneous solid cylinder 2 ft in diameter is given an angular velocity of 36 rad per sec and placed with its axis horizontal on a horizontal plane where the coefficient of friction is 0.24. If the cylinder started without translational velocity, what was its velocity when slipping ceased? *Ans.* 12 ft per sec.

26. In Prob. 25 find the time until slipping ceases, and the displacement during this time. Find the energy dissipated by friction and check the problem by an energy balance.

27. A 16-lb disk 3 ft in diameter is mounted to turn freely about its center in a vertical plane. A slender 48-lb bar 3 ft long turns freely in a vertical plane about a hinge 2.5 ft horizontally from the center of the disk. The disk carries at its edge a small smooth pin on which the bar rests in a position of unstable equilibrium. The disk is started slightly with the pin at its edge moving downward toward the hinge at the lower end of the bar, and the bar follows the pin downward sweeping across the front of the disk. Find the two angular velocities as the bar passes through the horizontal position ($g = 32$).

Ans. 2.84, 4.26 rad per sec per sec.

28. Find the components of the reactions at the two hinges of Prob. 27 as the bar passes through the horizontal position. (Let $g = 32$.)

Ans. Disk, $V = 42.9$, $H = 0$, bar, $V = 13.1$ (downward), $H = 40.9$.

29. Suppose b of Fig. 149 (page 141) to be 4 ft and the bar AB to be a slender uniform bar 3 ft long weighing 12 lb. Let the mass P be 40 lb and find the tension in the cord and the components of the hinge reaction as the bar AB passes through the horizontal position after having been released from the vertical position with B downward. (The velocity of P at this point is zero so that all the kinetic energy is in the bar. The acceleration of B toward A is $3\omega^2$, but *this is not the acceleration of P.* The acceleration of P is the second derivative with respect to time of the distance BC when $\theta = 0$).

Ans. 3828 lb tension in the cord.

Example 2

A mass of 25 lb slides down a rod in a vertical plane. The path of the center of gravity of the mass follows the curve $3y = x^2$. If the mass starts from rest at $y = 12$ and slides without friction, what is the total reaction of the rod when $y = \frac{4}{3}$? (See Art. 141.)

$$\frac{dy}{dx} = \frac{2x}{3}$$

When

$$y = \frac{4}{3} \qquad x = 2 \qquad \frac{dy}{dx} = \frac{4}{3}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{125}{27}$$

$$\frac{d^2y}{dx^2} = \frac{2}{3}$$

$$\rho = \frac{125}{27} \times \frac{3}{2} = \frac{125}{18}$$

$$N = \frac{25}{g} \times \frac{v^2}{\rho}$$

$$\frac{v^2}{2g} = 12 - \frac{4}{3} = \frac{32}{3} \qquad \frac{v^2}{g} = \frac{64}{3}$$

$$N = \frac{25 \times 64}{3} \times \frac{18}{125} = 76.8 \text{ lb}$$

$$Q = 76.8 + 25 \times 0.6 = 91.8 \text{ lb}$$

Problems

30. In the example above, find the total pressure at the lowest point.

Ans. $Q = 400 + 25 = 425$ lb.

31. A mass of W lb slides down a frictionless rod in a vertical plane from $y = 34.5$ ft to $y = 4.5$ ft. The rod is bent into the form of a rectangular hyperbola $xy = 27$. Find the total normal pressure.

Ans. $Q = 1.92W + 0.8W = 2.72W$

CHAPTER 17

MOMENTUM

165. Impulse and Momentum. *Rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.* This is Newton's second law of motion. It is expressed mathematically by

$$P\,dt = k\,d(mv) \tag{1}$$

Assuming, as is usually the case with engineering problems, that the mass of the body under study is constant, and selecting units so that the proportionality constant k is unity. Eq. (1) becomes

$$P\,dt = m\,dv \tag{2}$$

$$\int P\,dt = m \int dv \tag{3}$$

$$\int P\,dt = m(v_2 - v_1) \tag{4}$$

The term $m\int dv$ or the quantity mv is called *momentum.* The term $\int P\,dt$ is called the *impulse* of the force. When the force is constant during a time t the impulse is Pt. Equation (3) shows that the change of momentum is proportional to the impulse. The change in momentum is equal to the impulse in any system in which force $= ma$. If the force is in pounds, the mass in Eq. (4) must be given in slugs. If the force is in dynes, the mass is in grams. If the mass is in pounds, the force is in poundals,

$$\int P\,dt = \frac{W}{g}(v_2 - v_1) \tag{5}$$

in which P is the force in pounds, W is the mass in pounds, v is the velocity in feet per second, and $g = 32.174$.

When an impulse is given by a blow, as when an object is struck by a hammer, the force is large and the time is very short. Such an impulse is frequently called an *instantaneous impulse. Sudden impulse* is a better name. Suppose that a ball moving with a velocity of 100 feet per second strikes a wall and bounds back with a velocity of nearly 100 feet per second. Suppose that the compression of the ball amounts to 0.5 inch. If the force were uniform during the time of contact, the average velocity would be 50 feet per second while the ball is coming to rest. The time required to bring the ball to rest would be

$$\frac{1/24}{50} = 1/1200 \text{ sec}$$

The entire time of contact would be about 1/600 second. If the pressure varies as the displacement, instead of being uniform, the motion during contact is simple harmonic. The time required to complete a semicircle of ½-inch radius at a speed of 100 feet per second is $\pi/2400 = 0.0013$ second. When a hammer strikes an anvil, the deformation is much smaller and the time of contact is less. In general, the duration of a so-called instantaneous impulse is a few ten-thousandths of a second.

It is sometimes stated that an instantaneous impulse is completed before the body which is struck starts to move. This statement is incorrect. The full velocity is attained at the end of the impulse. The time, however, is so short that the actual displacement is small. In the case of a galvanometer needle, which is actuated by the discharge of a condenser, the needle has attained its maximum velocity at the time when the current becomes negligible. The deflection, on the other hand, is so small that the moment arm of the impulse is practically constant throughout the entire impulse.

Momentum is the product of a scalar (mass) and a vector (velocity) and is therefore a vector quantity and has direction as well as magnitude. It must not be confused with kinetic energy, which is a radically different kind of quantity. It is true that a body cannot have one without the other, but a small body with high speed can have much more kinetic energy than a large body with low speed which has a larger momentum. In a collision, the body with the large momentum is the dominant one. Momentum is an extremely useful concept in analytical mechanics, especially in fluid mechanics, as it often facilitates the writing of exact equations in situations where a detailed account of the forces and times involved is impossible.

Momentum is the most fundamental quantity of mechanics (and possibly of the universe). It is always completely conserved and never changes forms as does energy. While the momentum of a body may seem to have been created or destroyed by force and time, a closer view *will reveal that momentum* has *merely* been *transferred* from one body or system to another.

Problems

1. A mass of 64.348 lb has its velocity changed from 10 ft per sec east to 40 ft per sec east in 5 sec. Find the impulse. *Ans.* 60 lb-sec.
2. Solve Prob. 1 if the velocity changes from 10 ft per sec east to 40 ft per sec west. *Ans.* 100 lb-sec.
3. A force varies directly as the square of the time and is 2 lb when the time is 1 sec. Find the impulse during the interval from 1 to 4 sec. *Ans.* Impulse = 42 lb-sec.

166. Action and Reaction. When force acts on a body, the force must come from some other body. As stated by Newton's third law, action and reaction are equal. The force exerted by the second body on the first body is equal and opposite to the force exerted by the first body on the second body. Since the forces are equal, the impulses are equal, and, consequently, the change of momentum of one body is equal and opposite to the change of momentum of the other.

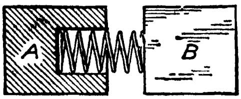

FIG. 314.

In Fig. 314, A and B are two bodies. Suppose that A exerts a force on B by means of a spring which forms a part of A. The change of momentum of A toward the left is equal and opposite to the change of momentum of B toward the right. Suppose that A with the attached spring weighs 5 pounds and B weighs 4 pounds, and suppose that both are stationary. When the spring is released, suppose that B receives a velocity of 10 feet per second toward the right. Its change of momentum is 40. At the same time, A receives a velocity of 8 feet per second toward the left. Its change of momentum is -40. The total change of momentum of the two bodies considered as a single system is zero. It is impossible to change the total momentum of two bodies by means of any force exerted between them. Any change in the momentum of a body is at the expense of an equal and opposite change in the momentum of some other body or bodies. When a man jumps upward into the air, he kicks the earth in the opposite direction with equal momentum. While he is in the air, the attraction of gravity draws him toward the earth and, likewise, draws the earth toward him. Considered as one body, the center of gravity of the earth and the man moves on unchanged.

If m_1 is the mass of one body and v_1 is its velocity, m_2 is the mass of a second body and v_2 is its velocity, and if a force acts between the bodies which changes the velocity of m_1 from v_1 to v'_1 and changes the velocity of m_2 from v_2 to v'_2,

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \qquad (1)$$

Since m and v occur in each term of Eq. (1) and force is not a factor, any units of mass and velocity may be used.

Example

A mass of 20 lb moving east 30 ft per sec overtakes a mass of 12 lb moving east 10 ft per sec. After collision, the bodies move together with the same velocity. Find that velocity.

The total momentum east is

$$\begin{aligned} 20 \times 30 &= 600 \\ 12 \times 10 &= \underline{120} \\ &\ 720 \end{aligned}$$

The total mass is 32 lb. If v is the final velocity,

$$\begin{aligned} 32v &= 720 \\ v &= 22.5 \text{ ft per sec east} \end{aligned}$$

Problems

1. A man weighing 150 lb jumps horizontally with a speed of 10 ft per sec from a boat weighing 200 lb. If the boat was initially at rest and if the friction of the water is neglected, with what velocity does the boat move in the opposite direction? *Ans.* 7.5 ft per sec.
2. A man weighing 150 lb starts from rest on a small barge, acquires a velocity of 24 ft per sec with reference to the ground in 3 sec, and jumps off. If the barge weighs 1200 lb, what is its final velocity with respect to the ground? How far does the man run with respect to the ground? How far with respect to the barge?
3. A shot weighing 10 lb is fired from a gun weighing 800 lb. The velocity of the shot relative to the earth is 1200 ft per sec. Find the velocity of the gun's recoil. Find the kinetic energy of the shot and of the gun.
   Ans. v = 15 ft per sec; kinetic energy of shot = 223,780 ft-lb; kinetic energy of gun = 2797 ft-lb.
4. When a shot is fired from a gun, show that the kinetic energy of the shot is to the kinetic energy of the gun as the weight of the gun is to the weight of the shot.
5. A 4-lb mass moving east 60 ft per sec meets a 5-lb mass moving west 20 ft per sec. After collision, the 5-lb mass moves east 20 ft per sec. By means of the total momentum find the velocity of the 4-lb mass. *Ans.* v = 10 ft per sec east.
6. If the 5-lb mass of Prob. 5 moves east 40 ft per sec after collision, what is the velocity of the 4-lb mass? *Ans.* v = 15 ft per sec west.
7. The bodies of Prob. 5 collide obliquely. After collision the 5-lb mass moves northeast with a velocity of 30 ft per sec. What is the velocity of the 4-lb mass?
   Ans. The 4-lb mass has a velocity component east of 8.48 ft per sec and a component south of 26.51 ft per sec.

167. Collision of Inelastic Bodies. When inelastic bodies collide, they remain together after collision and move with the same velocity. If m_1 and m_2 are the masses, v_1 and v_2 their respective velocities before collision, and v is their common velocity after collision. the momentum equation is

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v \tag{1}$$

When inelastic bodies collide and adhere, the total momentum of the two (vector sum) is conserved and becomes the momentum of the combined mass. If the paths of the two bodies are collinear, the center of mass of the combined body will follow that line. If the paths are parallel but not collinear, the center of gravity of the combined mass will follow a parallel path between the two original paths and the combined mass will have an angular velocity. In any case there will be a loss of kinetic energy (converted to heat) and this loss is referred to as an *impact loss.*

Problems

1. A mass of 50 lb moving east 24 ft per sec overtakes a mass of 30 lb moving east 16 ft per sec. After collision the bodies move together. Find their common velocity. *Ans.* 21 ft per sec.
2. In Prob. 1, what part of the kinetic energy is lost? *Ans.* 3.29 per cent.
3. Solve Probs. 1 and 2 if the 30-lb mass is moving west instead of east. *Ans.* 9 ft per sec; 82.2 per cent.
4. A gun is hung in a horizontal position on a vertical support which is hinged at the top as shown in Fig. 315. The gun and support weigh 800 lb and their center of gravity is 12 ft below the hinge. When a 10-lb shot is fired from the gun, the reaction swings the support through an arc of 45°. Find the velocity of the shot. *Ans.* $v = 1203$ ft per sec.

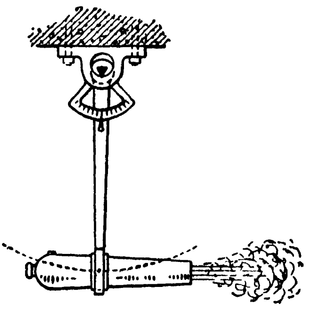

FIG. 315.

5. A soft wood block weighing 5 lb is suspended by vertical cords 5 ft in length. A bullet weighing 0.02 lb is fired horizontally into the block and remains embedded in it. The block swings through a vertical angle of 35°. Find the velocity of the bullet.
6. A 10-lb body moving east at 20 ft per sec collides with a 5-lb body moving west at 25 ft per sec and north at 15 ft per sec. If the bodies adhere, what is their final velocity? *Ans.* 7.07 ft per sec north 45° east.

168. Collision of Elastic Bodies. When two elastic bodies collide, both bodies suffer a change of form. Their centers of mass continue to approach each other for a little time and both bodies are compressed at the surface of contact. If one of the bodies is fixed so as to be practically stationary, the other body will come to rest and then rebound. If both bodies are free to move, there comes a time when each is stationary relative to the other, and both are moving with the same velocity relative to the earth. Figure 316 represents three stages of the collision of two bodies. In Fig. 316,I, the bodies are moving toward each other at the beginning of contact. In Fig. 316,II, the centers O_1 and O_2 are nearest to each other. The bodies are stationary relative to each other and are moving with a velocity v relative to the earth. It is assumed that the momentum of m_1 is the greater, the common velocity is, therefore, toward the right. The interval from the beginning of collision to the condition of Fig. 316,II is called the *compression stage* of the collision. The common velocity at the end of the compression stage is found in the same way as if the bodies were inelastic.

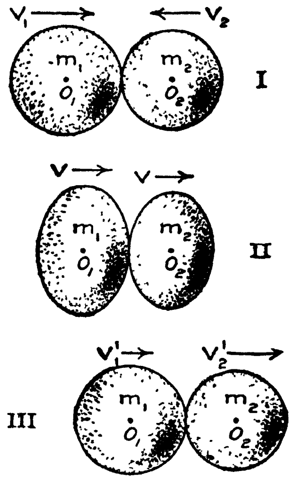

FIG. 316.

After the bodies have reached the end of the compression stage, their elasticity causes them to separate. The interval from the end of the compression stage to the time when the bodies cease to be in contact with each other is called the *restitution stage*. Figure 316,III shows the bodies at the end of the restitution stage.

If the bodies are perfectly elastic, the impulse during restitution is equal to the impulse during compression and the change of momentum of each body during restitution is equal to the change during compression. With imperfectly elastic bodies the change of momentum during restitution is less than the change during compression. The ratio of the impulse during restitution of the impulse during compression is called the *coefficient of restitution.* When the bodies are perfectly elastic, the coefficient of restitution is unity. The coefficient of restitution is represented by the letter e.

Since the masses of the bodies do not change as a result of the collision, the *change of velocity* of either body during the compression stage divided into the change of velocity of the same body during the restitution stage equals e. The original definition of e by Newton was *the ratio of the velocity of departure of the two bodies to their velocity of approach,* both measured along a line perpendicular to the common tangent at the point of collision. The coefficient of restitution is an experimental factor for any two bodies, depending upon their shapes and materials. It is not independent of the striking velocity but can be assumed so for small velocities.

Problems

1. A body is thrown against a fixed wall with a speed of 50 ft per sec and rebounds with a speed of 38 ft per sec. Find the coefficient of restitution. *Ans.* $e = 0.76$.
2. A ball is dropped 16 ft upon a horizontal surface and rebounds 9 ft. Find the coefficient of restitution. *Ans* $e = 0\ 75$.
3. A body is dropped 12 ft on a horizontal surface. If the coefficient restitution is 0 8, how high will it rise?
4. A body strikes a perfectly smooth plane with a velocity of 50 ft per sec at an angle of 30° with the plane. If the coefficient of restitution is 0.8, what will be its velocity when it leaves the plane? *Ans.* $v = 47.7$ ft per sec at 24°47′ with the plane.
5. What must be the coefficient of friction between the body and the plane of Prob. 4 so that the angle of incidence will equal the angle of reflection? (The vertical impulse multiplied by the coefficient of friction equals the horizontal impulse if slipping takes place during the entire time of contact) *Ans* 0.193.

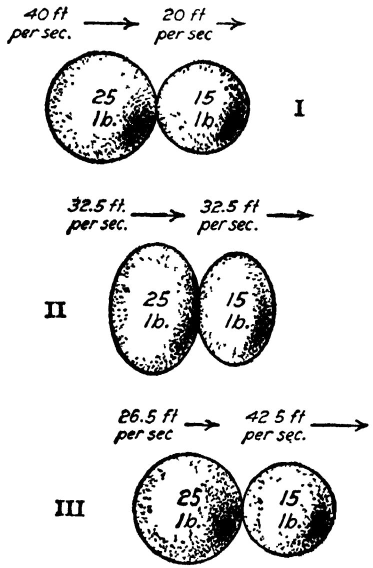

FIG. 317.

The behavior of a pair of elastic bodies during collision is best studied by a numerical example rather than by the use of literal formulas. Figure 317,I represents a mass of 25 pounds moving east with a velocity of 40 feet per second, which is overtaking a mass of 15 pounds moving east with a velocity of 20 feet per second. Figure 317,II shows the two bodies at the end of the compression stage. It may be assumed that the bodies are photographed and that the plate is shifted vertically between exposures. The initial momentum is

$$25 \times 40 + 15 \times 20 = 1300.$$
$$v_c = {}^{1300}\!/_{40} = 32.5 \text{ ft per sec}$$

As a condition of the problem, it will be assumed that the coefficient of restitution is 0.8. The change of velocity of the 25-pound mass during compression is $32.5 - 40 = -7.5$ feet per second. The change of velocity of this mass during restitution is

$$0.8\,(-7.5) = -6 \text{ ft per sec}$$

The final velocity of the 25-pound mass is

$$v = 32.5 - 6 = 26.5 \text{ ft per sec east}$$

The change of velocity of the 15-pound mass during compression is 12.5 feet per second. Its change during restitution is 10 feet per second. The final velocity of the 15-pound mass is

$$v = 32.5 + 10 = 42.5 \text{ ft per sec east}$$

The results may be checked by means of the final momentum,

$$\begin{array}{lcl} 25 \times 26.5 & = & 662.5 \\ 15 \times 42.5 & = & 637.5 \\ \hline \text{Total momentum} & = & 1300 \end{array}$$

Problems

6. A mass of 40 lb moving east 30 ft per sec overtakes a mass of 20 lb moving east 15 ft per sec. The coefficient of restitution is 0.8. Find the common velocity at the end of the compression. Find the final velocity of each mass. Check the final momenta. *Ans.* 25 ft per sec; 21 ft per sec; 33 ft per sec.

7. Solve Prob. 6 if the initial velocity of the 20-lb mass is 15 per sec west. *Ans.* Check is $40 \times 30 - 20 \times 15 = 40 \times 3 + 20 \times 39$.

8. A mass of 20 lb moving east 40 ft per sec meets a mass of 5 lb moving west 80 ft per sec. After collision the 5-lb mass moves east 88 ft per sec. Find the velocity of the 20-lb mass and the coefficient of restitution.

9. A mass of 10 lb moving east 30 ft per sec meets a mass of 10 lb moving west 12 ft per sec. After collison the mass which was moving east 30 ft per sec is moving west 12 ft per sec. Find the coefficient of restitution. Check by Newton's definition of the coefficient of restitution.

10. If two perfectly elastic bodies of equal mass collide, show that they exchange velocities.

11. What per cent of the total energy is lost in Prob. 1? What per cent is lost in Prob. 2? *Ans.* 42.24 per cent; 43.75 per cent.

12. What per cent of the total energy is lost in Problem 7? *Ans.* 24.0 per cent.

13. A baseball moving 100 ft per sec is struck by a bat moving 60 ft per sec in the opposite direction. If the bat weighs six times as much as the ball and the coefficient of restitution is 0.9, with what velocity will the ball leave the bat? *Ans.* 160.6 ft per sec.

(In the remainder of the problems in this list, use $g = 32$.)

14. A 4-lb body with unknown velocity struck an 8-lb body and stopped dead. The bodies were on a horizontal plane where the coefficient of friction was 0.25. If the 8-lb body was displaced 4 ft, what was the coefficient of restitution and the initial velocity of the 4-lb body? *Ans.* $e = 0.5$.

15. A 5-lb ball suspended as a pendulum 4 ft long was released from a position where its cord was taut and horizontal. It swung downward through 90°, struck a 5-lb block, and knocked it from the edge of a table 4 ft high. If the block struck the floor 7 ft from the table edge, what was the coefficient of restitution?

16. A ping-pong ball rolled off a table 4 ft high, struck the floor 8 ft from the table, and moved 12 ft farther on the first bounce. Find the value of the coefficient of restitution. *Ans.* $e = 0.75$.

17. If the ball of Prob. 16 continued to bounce on a horizontal floor, show that each bounce took it e times the preceding one, neglecting air resistance. If the ball rolled from the table, is it reasonable to neglect friction between the ball and the floor?

18. A 20-lb mass sliding on a horizontal floor has a velocity of 20 ft per sec. It slides 9 ft, strikes a similar body weighing 32 lb, and adheres. How far will the two bodies go before coming to rest, if $\mu = 0.25$ for both bodies?

19. A 6-lb block on a smooth horizontal table is attached to a cord which runs horizontally through a hole in a 24-lb block at rest on the same table, then over a small smooth pulley and down to a suspended block weighing 10 lb. The clearance between the blocks on the table is 2.5 ft. If the 6-lb block is released, how far will it move in the first second if it adheres to the 24-lb block on striking? *Ans.* 5.5 ft.

20. A 24-lb shot propelled by a 4-lb powder charge is given a muzzle velocity of 1600 ft per sec from a gun tube weighing 800 lb. Assume the forward component of velocity of the gas (all the powder) to be 6000 ft per sec and find the velocity of recoil.

21. A muzzle brake is a device which catches most of the blast of a gun and directs the gases toward the rear. Solve Prob. 19, assuming a muzzle brake catches all the gas and discharges it toward the rear at an angle of 30° with the axis of the tube and a velocity of 6000 ft per sec. *Ans.* 22 ft per sec.

22. An 8-lb body with unknown velocity strikes squarely against a 4-lb body on a horizontal plane. The 4-lb body was initially at rest and was driven 12.25 ft by the blow. The 8-lb body went on in the same direction $1\frac{9}{16}$ ft before coming to rest. If the coefficient of friction was 0.25 for both bodies, what was the coefficient of restitution? Check by finding the initial velocity of the 12-lb body and comparing the velocity of approach with the velocity of departure.

169. Moment of Momentum.

If a body of mass m is moving with linear velocity v, its momentum is mv. If O (Fig. 318) is an axis perpendicular to the plane of the paper and if r is the length of the line perpendicular to this axis and to the line of motion of the mass m, the product of the momentum of m multiplied by the length r is the *moment of momentum* of m with respect to the axis. When a body is rotating about an axis, the linear velocity of an element dm at a distance r from the axis is given by the equation

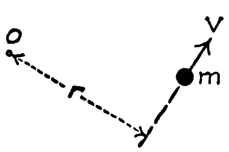

Fig. 318.

$$v = r\omega \qquad (1)$$

The momentum of this element is $r\omega \, dm$, and the moment of momentum with respect to the axis of rotation is $r^2\omega \, dm$. The total moment of momentum is

$$\omega \int r^2 \, dm = \omega I \tag{2}$$

The moment of momentum with respect to an axis of a body which is rotating about that axis is the product of the moment of inertia of the body multiplied by its angular velocity. A comparison of moment of momentum with momentum shows that moment of inertia replaces mass, and angular velocity replaces linear velocity.

To change the momentum of a body, force must be applied from an outside body. Likewise, to change the moment of momentum of a body, torque must be applied from some outside body or system of bodies. From Formula XXXVIII,

$$T = I\alpha = I \frac{d\omega}{dt} \tag{3}$$

$$\int T \, dt = I \int d\omega = I\omega \tag{4}$$

If the torque is constant,

$$Tt = I\omega \tag{5}$$

The product Tt may be called the *torque impulse.* The vector quantity $I\omega$, moment of momentum, is also called *angular momentum* and sometimes *spin momentum.* The three terms all have exactly the same meaning. When the quantity is represented by a vector, the vector is drawn parallel to the axis of rotation and the right-hand screw

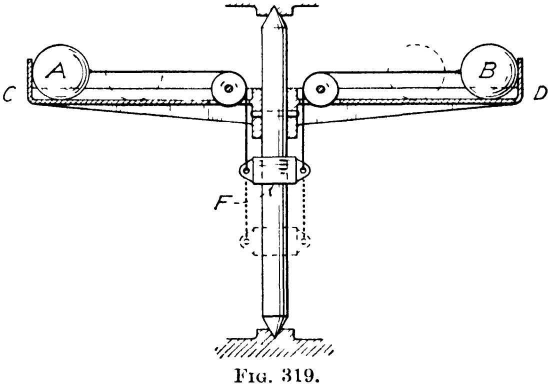

FIG. 319.

convention is usually used for its direction. The moment of momentum of a rotating body may be changed only by the application of torque impulse (or by removing from or adding to the body elements having moment of momentum). The angular velocity may be changed by changing the configuration of the body in such a way as to alter its moment of inertia. Figure 319 shows a frame arranged to rotate about a vertical axis. The frame carries two equal bodies *A* and *B*. These bodies are connected to cords which run over pulleys and are attached to the sleeve *F* on the vertical shaft. Suppose that the frame is rotating with an angular velocity ω when *A* and *B* are in the positions shown in the figure. If the sleeve *F* is now pulled down by some outside force, or by its own weight, and draws *A* and *B* in toward the axis, the moment of inertia of the system is reduced.

Forces which exert no torque on a system cannot change the moment of momentum. When these forces do work, however, they must change the kinetic energy. When the bodies of Fig. 319 are drawn toward the axis, the moment of inertia is reduced and the angular velocity is correspondingly increased. Since the energy varies as the square of the angular velocity, there is an increase in the total kinetic energy, which comes from the work done on the sleeve.

Example

A frame similar to Fig. 319 carries two spheres, each of which weighs 64.348 lb and has a radius of gyration of 0.5 ft. The frame weighs 96.522 lb and has a radius of gyration of 1 ft. When the masses are each 2 ft from the axis the system has an angular velocity of 8 rad per sec. The sleeve is released and the centrifugal force from *A* and *B* lifts it until the spheres are stopped with each center 3 ft from the axis of the vertical shaft. If there is no friction loss, what is the angular velocity?

The moment of inertia, in slugs feet squared, is

$$\begin{aligned} I_2 &= 3 \times 1 + 4 \times 0.25 + 4 \times 4 = 20 \\ I_3 &= 4 + 4 \times 9 = 40 \\ 20 \times 8 &= 40\omega_3 \\ \omega_3 &= 4 \text{ rad per sec} \end{aligned}$$

The kinetic energy is

$$U_2 = \frac{20}{2} \times 64 = 640 \text{ ft-lb}$$

$$U_3 = \frac{40}{2} \times 16 = 320 \text{ ft-lb}$$

To find the maximum weight of the sleeve in order that it can be lifted by the centrifugal force,

$$W = m\bar{r}\omega^2 = 4 \times 2 \times 64 = 512 \text{ lb}$$

in which W is the weight which is balanced at 2 ft. At 3 ft from the axis

$$W = 4 \times 3 \times 16 = 192 \text{ lb}$$

Problems

1. Solve the example above when the spheres are pulled in till their centers are each 1 ft from the axis. Find the additional pull on the sleeve if it weighs 192 lb. *Ans.* $\omega_1 = 20$ rad per sec; $U_1 = 1600$ ft-lb; pull $= 1408$ lb.
2. A circular disk 2 ft in diameter and weighing 4 lb rotates about a vertical axis with an angular velocity of 6 rad per sec. A second disk 1 ft in diameter and weighing 16 lb is coaxial with the first disk. This second disk is dropped a short distance upon the first disk and adheres to it. If the second disk is initially stationary, what is the common velocity of the two when they rotate together? *Ans.* $\omega = 3$ rad per sec.
3. Solve Prob. 2 if the axis of the second disk is 6 in. from the axis of rotation.
4. Solve Prob. 2 if the second disk has an initial velocity of 10 rad per sec in the same direction as the first disk.
5. If the 16-lb disk of Prob. 2 is slightly dished so that it is in contact with the lower disk only at its periphery and the coefficient of friction between the disks is 0.4, what is the time from the beginning of the drop until the two disks are turning together?

   The sudden downward impulse of the upper disk on the lower at the instant of contact is equal to the weight of the upper disk multiplied by the time of falling, so that the height from which it is dropped makes no difference. The average vertical impulse is 16 lb. The torque impulse from friction is

$$(16 \times 0.4 \times 0.5)t = \frac{16 \times (0.5)^2}{2g} 3 \qquad t = \frac{1.875}{g} \text{ sec}$$

   Check the result by a consideration of the torque and change of velocity of the 4-lb disk.
6. Solve Prob. 5 if the disks are assumed flat with uniform pressure all over the area of contact.
7. In Prob. 2, suppose the 16-lb disk to be mounted on a frictionless shaft and to be brought edge to edge against the rotating 4-lb disk and held until slipping ceases. What then would be the velocities of the two disks? *Ans.* 4-lb disk, 3 rad per sec; 16-lb disk, 6 rad per sec.

170. Force of Steady Streams. In any situation where a steady stream impinges against a body, stationary or moving uniformly, the equation $\int F\,dt = \int M\,dv$ integrates into

Force × time = mass × change of velocity

i.e., the force is constant and the time may be any desired finite time. The mass is the mass which had its velocity changed during that time and the change of velocity means the change in the velocity component in the direction of the force. The proposition is especially useful in fluid flow. When a stream of fluid strikes a flat plate as in Fig. 320 it spreads out and goes off in all directions with nearly the same velocity with which it struck, but the velocity in the direction of flow is completely stopped if the plate is not moving. The intensity of pressure is highest at the point directly in front of the middle of the stream and diminishes outward from that point in a manner difficult to determine, but the *total pressure* can be determined very simply from the equation $Ft = Mv$.

Example

Salt water weighing 64.4 lb per cu ft is squirted from a nozzle squarely against a large flat plate. If the velocity of the jet is 100 ft per sec and its cross-section area is 0.1 sq ft, what is the pressure against the plate?

Let $t = 1$ sec; $M = \dfrac{100 \times 0.1 \times 64.4}{g}$; $v = 100$.

$$F = 20 \times 100 = 2000 \text{ lb}$$

(The result seems astonishingly large. The velocity is not unusual in ordinary streams from fire hoses, but the area is fifty times that of a large fire stream.)

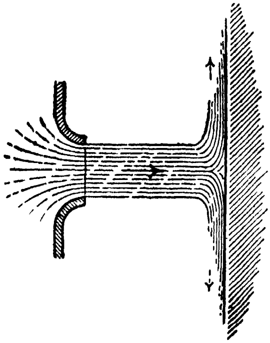

FIG. 320.

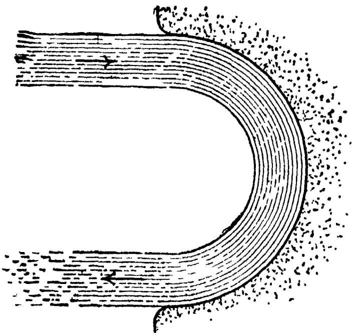

FIG. 321.

If the stream strikes a curved surface which changes its direction 180°, as shown in Fig. 321, the change in velocity is the outgoing velocity minus the incoming velocity, and since the incoming velocity is positive and the outgoing one is negative, this amounts to a change in the direction of the outgoing velocity equal to the numerical sum of the velocities. With the directions taken as indicated above, the equation gives the force which the surface exerts on the stream to change its direction. The force which the stream exerts on the surface is, of course, equal and opposite. If the stream of the example is reversed instead of being merely stopped, the change of velocity would be 200 and the force 4000 pounds. It is not possible to have the exit velocity equal to the entrance velocity. Friction slows the stream down and it comes out with a larger cross section and a slower velocity, but the exit velocity can easily exceed 90 per cent of the entrance velocity under good conditions.

Figure 322 shows the cross section of one bucket of a Pelton wheel. The stream strikes the splitter and divides, each portion having its direction changed nearly 180°. A series of buckets is arranged around the wheel so that all the stream goes into one or another of the buckets. The greatest efficiency is attained when the buckets move at one-half the velocity of the oncoming stream.

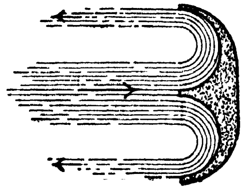

FIG. 322.

Problems

1. Solve the example if the plate is moving at 50 ft per sec in the direction of the stream. *Ans.* $F = 500$ lb.
2. Suppose the stream of the example is completely reversed by a bucket as in Fig. 322 with the outgoing velocity the same as the ingoing, relative to the bucket. Find the force exerted on the bucket as it moves 50 ft per sec in the direction of the stream. *Ans.* 1000 lb.
3. Find the average tangential force on the wheel of which the bucket of Prob. 2 is typical. *Ans.* $F = 2000$ lb.
4. The tangential force on the wheel times the peripheral velocity is the *power*, or rate of doing work. Compare the power of the wheel of Prob. 3 with the kinetic energy of the water entering in 1 sec to show that under the assumptions made the efficiency is 100 per cent.
5. Solve Probs. 2 and 3 and find the efficiency if the exit velocity from the buckets is 80 per cent of the entering velocity. *Ans.* 900 lb; 1800 lb; 90 per cent.
6. A chain is held by one end with the other end just touching a horizontal plane. If the chain is released from this position, what will be the maximum pressure on the plane during the chain's fall? *Ans.* Twice the weight.

This result can be checked by dropping the chain on a small platform scale set at twice the weight of the chain. Just as the last link strikes, the poise can be seen to rise slightly. (It is necessary to hold the lower end slightly above the platform to compensate for the height of the pile of chain the last link strikes.)

171. Rotation Caused by Eccentric Impulse.

If a body is given an impulse by a force acting along a line passing through the center of gravity, there is no moment about the center of gravity. The body then has no torque impulse and the velocity imparted by the impulse is purely linear. If exactly the same impulse is given the body along a line not passing through the center of gravity, the center of gravity of the body will be given the same linear velocity as before and there will be also an angular velocity which can be found by Eq. (5) of Art.

169 just as though the body had been hinged at the center of gravity. The *work* done by the impulse is greater than in the first case since the motion of the point at which the impulse was applied moved, during the impulse, the same distance the center of gravity moved plus an angle multiplied by the distance from the center of gravity to the point.

Example 1

'A slender rod 3 ft long weighing 8.05 lb was placed on a smooth floor and struck a horizontal blow perpendicular to the bar, which imparted to the center of the bar a velocity of 12 ft per sec. If the blow was at one end of the bar, what was the impulse and what angular velocity was imparted to the bar?

$$\text{Impulse} = \frac{8.05 \times 12}{g} = 3 \text{ lb-sec}$$

$$\text{Torque impulse} = 3 \times 1.5 = 4.5 \text{ lb-ft-sec}$$

$$4.5 = \frac{8.05 \times 3^2}{12g}\,\omega \qquad \omega = 24 \text{ rad per sec.}$$

Example 2

The bar of Example 1 is hinged at its mid-point and placed in the horizontal position. A small ball with a mass of 0.5 lb is dropped 4 ft onto the end of the bar and rebounds 1 ft. Find the angular velocity imparted to the bar and find the coefficient of restitution.

Velocity at the end of the fall (just before striking), taking $g = 32$ ft per sec per sec, = 16 ft per sec

Velocity upward after rebound = 8 ft per sec

Change of velocity = 24 ft per sec

$$\text{Impulse of bar on ball} = \frac{24 \times 0.5}{g}$$

$$\text{Torque impulse of ball on bar} = \frac{24 \times 0.5 \times 1.5}{g} = \frac{8 \times 3^2\omega}{12g}$$

From which $\omega = 3$ rad per sec.

Downward velocity of the end of the bar after restitution $= 1.5\omega = 4.5$

Upward velocity of the ball after restitution $= 8.0$

Velocity of departure $= 12.5$

Velocity of approach = 16 ft per sec; $e = \frac{12.5}{16} = 0.781$

Example 3

If the ball of Example 2 adheres to the bar instead of rebounding, what will be the common velocity?

Solution 1: The bar of Example 2 was initially at rest. Therefore, the final velocity, 3, $= (1 + e)\omega_c$; $\omega_c = 3/1.781$ rad per sec.

Solution 2: The linear momentum of the ball just before striking multiplied by its distance from the hinge is its moment of momentum with respect to the hinge, and this quantity is equated to the angular momentum of the bar and the ball combined.

Moment of inertia of bar and ball, in pounds feet squared, $= \dfrac{8 \times 3^2}{12} + 0.5 \times 1.5^2$
$= 7.125$

Moment of momentum of ball, in pounds feet squared per second,
$= 0.5 \times 16 \times 1.5 = 12$

Angular velocity after impact $= \dfrac{12}{7.125} = \dfrac{3}{1.781}$, as before.

Solution 3: The bar and the ball may be considered a single free body before as well as after getting together. Before getting together, the weight of the bar is balanced by the upward reaction at the hinge. The gravity force on the ball has a moment about the hinge equal to 0.5 × 1.5 lb-ft. The time of fall is 4 ft divided by average velocity $= \frac{4}{8} = 0.5$ sec. Equating the torque impulse to the moment of momentum, $0.5 \times 0.75 = \dfrac{7.125\omega}{g}$.

Problems

(*Let* $g = 32$.)

1. A slender 6-lb bar 2 ft long is hinged at the right end and rests in a horizontal position on a support near the left end. The left end is struck from below by a small hammer weighing 3 lb and moving at 8 ft per sec. If the coefficient of restitution is 0.5, how high will the bar rise becore dropping back to place and what will the components of the hinge reaction be just after restitution?
Ans. $h = 0.54$ ft; $V = 1.5$ lb; $H = 2.43$ lb.
2. A wheel weighing 8 lb has a radius of gyration of 1.732 ft. It is mounted to turn freely in a vertical plane, and a small 4-lb ball of soft clay is dropped 2 ft into a cup on the periphery of the wheel at the same height as the center of the wheel and 2 ft from this center. If the ball adheres, through what angle will the wheel turn before reversing? *Ans.* 203°34′.
3. A machine gun fires 150 bullets per minute into a heavy block mounted on frictionless rollers, and the block is held in place by a spring balance. What will be the reading on the spring balance if the bullets weigh 0.03 lb each and strike the block with a velocity of 2700 ft per sec? (Assume the block heavy enough to damp out the irregularities in the reading.)
4. A large wooden disk is mounted to turn freely on a central shaft and the stream of bullets from the gun of Prob. 3 is shot through the disk at a point 2 ft from the center, with the path of the bullets at right angles to the radii at the points struck and at an angle of 30° with the shaft. What velocity will be imparted to the disk in 4 sec if the disk has a moment of inertia of 160 lb-ft² and the bullets slow down to 1200 ft per sec as they pass through? (Assume the direction of the bullets unchanged as they go through.) Account for all the kinetic energy of the bullets.
5. A sphere is given a translational velocity of 12 ft per sec without angular velocity on a horizontal plane where the coefficient of friction is 0.2. Find how long the ball will take to acquire an angular velocity sufficient to make the point

of contact with the plane the instantaneous center. (This is known as the "bowling ball problem.") *Ans.* $t = 12/0.7g$.

6. An airplane weighing 6000 lb with a radius of gyration of 9 ft with respect to a vertical axis through its center is flying at 600 ft per sec and fires a stream of bullets as in Prob. 3 forward from a gun 6 ft from the center of the plane. Find how much the plane is slowed down and how much deflected in 10 sec.
7. A machine spindle running between two driven pulleys is clutched alternately to one and the other. The two pulleys run in opposite directions with the same speed ω. Find the amount of energy dissipated by the clutches for each reversal of the spindle. *Ans.* Four times the kinetic energy of the spindle.

CHAPTER 18

HARMONIC VIBRATION

172. Periodic Motion. A periodic motion is a motion which repeats itself exactly after a certain interval of time. This time interval is called the *period* of the motion and such a motion is loosely called *vibration*, whether it is linear or angular. Angular periodic motion is properly called *oscillation*. The projection onto a line (such as a diameter) of uniform circular motion is called *simple harmonic motion*, or sometimes just *harmonic motion*. In speaking of such motion a *vibration* is sometimes used as synonymous with a complete period, but in better usage a vibration is one-half a complete period, or the motion between any two instantaneous stops which the moving point or body makes as it goes back and forth. Simple harmonic motion is of great importance in engineering work, first because it is the type of motion produced by an ordinary spring acting on a mass and second because more complex periodic motions are usually composed of two or more simple harmonic motions of different periods and amplitudes.

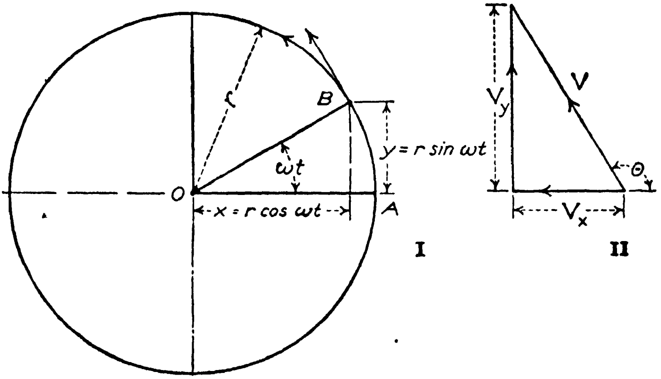

Fig. 323.

173. Components of Circular Motion. Figure 323 represents a particle which is moving on the circumference of a circle of radius r with an angular velocity ω. Its linear velocity is $r\omega$. The time required for the particle to move from A on the X axis to any point B is represented by t, and the angle AOB is ωt radians.

The coordinates of B are $x = r \cos \omega t$; $y = r \sin \omega t$. The velocity components are

$$v_x = \frac{dx}{dt} = -r\omega \sin \omega t \tag{1}$$

$$y_y = \frac{dy}{dt} = r\omega \cos \omega t \tag{2}$$

The components of the acceleration are found by differentiating the velocity components.

$$a_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt} = -r\omega^2 \cos \omega t \tag{3}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{dv_y}{dt} = -r\omega^2 \sin \omega t \tag{4}$$

When $t = 0$, $\cos \omega t = 1$ and $a_x = -r\omega^2$; $\sin \omega t = 0$; $a_y = 0$. The resultant acceleration when $t = 0$ (at A of Fig. 323) is $-r\omega^2$. It is along the X axis toward the center O. When $t = \pi/2$ $a_y = -r\omega^2$ and $a_x = 0$. At this position, then, the resultant acceleration is downward toward O.

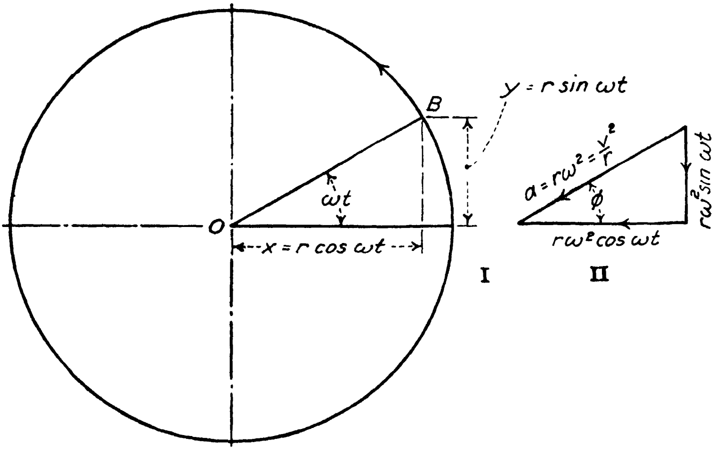

FIG. 324.

At any position, such as B of Fig. 324, the acceleration is the resultant of a_x and a_y. The acceleration vector diagram is shown in Fig. 324,II. The resultant acceleration is $r\omega^2$. The tangent of the angle ϕ which the resultant makes with a_x is $\frac{\sin \omega t}{\cos \omega t} = \tan \omega t$. Consequently, the acceleration at *any point B* of Fig. 324,I is along the line BO, which passes through the center of the circle. Since a_x and a_y are negative in the first quadrant, the resultant acceleration is downward toward the left. The resultant acceleration is always directed toward the center of the circle.

If v is the linear velocity of the point B,

$$\omega = \frac{v}{r} \qquad r\omega^2 = \frac{v^2}{r}$$

This result is the same as Formula XXVIII, the expression for the acceleration toward the center of curvature of a point moving along a curve. If the coordinates of the moving point B are factored out of Eqs. (3) and (4) and replaced by the symbols x and y, these equations become

$$a_x = -\omega^2 x \tag{5}$$

$$a_y = -\omega^2 y \tag{6}$$

Either the x or y component of uniform motion around a circle is, by definition, a simple harmonic motion. Equations (5) and (6) show that for such a motion the acceleration is opposite to the displacement and is directly proportional in magnitude to the displacement. Simple harmonic motion is caused by a force which acts on a mass and tends to restore it to a position of equilibrium when the intensity of the restoring force is proportional to the displacement of the mass from its equilibrium position. Such a force is exerted by a simple linear spring.

In the discussion above, it was assumed that time was measured from the instant when the body was at point A in Fig. 323. Equally simple expressions would have resulted if time had been measured from the instant the body passed the lowest point of the circle. This change would have given expressions for displacement and acceleration on X in terms of sines instead of cosines. If it is desired to measure time from some other instant, such as the time when the body passed point B, it can be done conveniently by letting the arc AB be symbolized by β. The coordinates of the moving point then become $x = r \cos (\omega t + \beta)$ and $y = r \sin (\omega t + \beta)$.

174. Force of a Spring. The deformation of a spring or other elastic body is proportional to the force. Figure 325 shows a helical spring which is stretched a definite amount d when a load of 1 pound is hung on the lower end. A load of 2 pounds would produce an elongation of $2d$. The forces on the spring are downward and the elongation is downward. The force with which the spring acts on the loads is upward, opposite to the deformation.

When a body is attached to a spring, there is some position at which it remains at rest. In this position, either the force of the spring is zero or the resultant of the force of the spring and the other forces is zero. Figure 326 represents a body on a flat strap spring. The plane of the paper is supposed to be horizontal in Fig. 326,I. The weight of the body tends to bend the spring in a vertical direction but not in a horizontal direction. Equilibrium exists, therefore, when the spring viewed from above appears straight. If the body is displaced horizontally either way from the position of equilibrium, the spring tends to move it back to that position with a force proportional to the displacement. Stated mathematically,

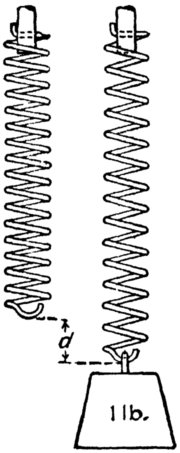

Fig. 325.

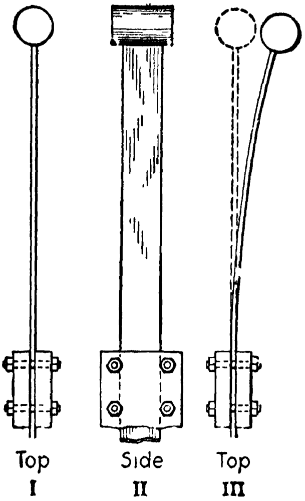

Fig. 326.

$$P = -Kx \tag{1}$$

in Fig. 325, if y is measured from the position of equilibrium of the mass hanging on the spring, the resultant of gravity and the spring force is a force restoring the mass to the position of equilibrium and

$$P = -Ky \tag{2}$$

In both these equations, P represents the force acting on the mass when it is displaced any distance from the equilibrium point and K is the external force required to hold the mass at unit distance from the equilibrium position. If the mass is not subjected to any other force, its acceleration at any place will be exactly that stated in Eqs. (5)

and (6) of Art. 173. The resulting motions will be simple harmonic, as will be seen from the more complete analysis following.

Problems

1. A springboard is deflected 7 in. downward by a load of 140 lb on the free end. What is the value of K? *Ans.* 240 lb per ft.
2. What is the deflection of the springboard of Prob. 1 when a boy weighing 120 lb stands on the free end? *Ans.* 0.5 ft.
3. The springboard of Prob. 2 with the boy on the end is moving up and down with a maximum deflection of 1 ft. Neglecting the mass of the board and assuming the boy to stand rigid, find the acceleration at the top and bottom points of the path. *Ans.* g upward at the bottom and down at the top.

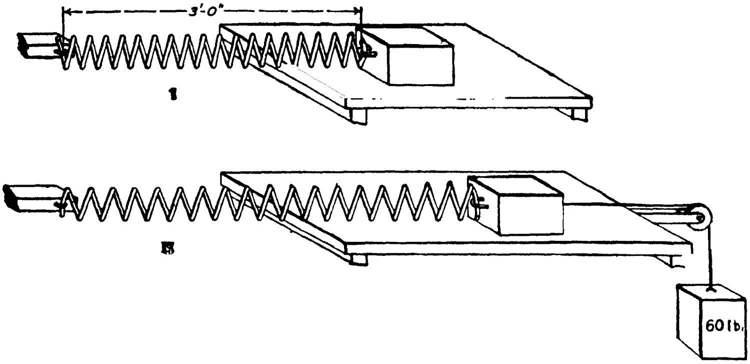

FIG. 327.

4. Figure 327,I shows a mass on a smooth horizontal plane attached to a horizontal spring of a natural length of 1 ft. The constant of the spring is 24 lb per ft and the mass weighs 72 lb. If the mass is drawn back and released, what will be its acceleration when the spring has a length of 5 ft? *Ans.* $2g/3$.
5. In Fig. 327,II, a cord is attached to the mass on the plane and run horizontally over a light, smooth pulley and down to a suspended mass of 60 lb. What is the total length of the spring when the system is in equilibrium? What will be the tension in the cord at the instant the spring has a total length of 8 ft after the 60-lb mass has been pulled down and released? *Ans.* 5.5 ft; 87.28 lb.

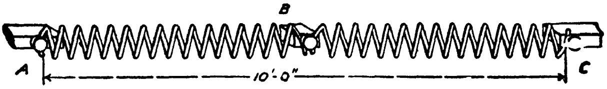

FIG. 328.

6. Figure 328 shows two springs attached to fixed points A and C, which are at the same level and are 10 ft apart. The natural length of each spring is 3 ft and a force of 60 lb stretches each spring 1 ft. A body B is fastened to the free ends of these springs. What is the resultant force on the body when it is displaced 1 ft from the position of equilibrium, and when it is displaced x ft from that position? *Ans.* $P = 120$ lb.

7. The two springs of Fig. 328 are replaced by a single spring of 6 ft natural length. A force of 30 lb stretches this spring 1 ft. The body is attached 4 ft from the left end of the stretched spring. What force will displace the body 1 ft in the direction of the length of the spring? What would be the constant if the body were attached 2 ft from the end of the stretched spring?
Ans. $P = 125$ lb; $K = 187.5$ lb.
8. Show that if a fraction c of its length is cut off from a uniform coil spring of constant K the constant of this part of the spring will be K/c.
9. A spring of constant 60 lb per ft is arranged with an initial stretch like the spring of Prob. 7. How far from the end to the point of attachment which gives a constant of 320 lb per ft? *Ans.* One-fourth of the length.

175. Potential Energy of a Spring. The work done by a force is the average force multiplied by the displacement. When the force varies as the displacement, the average force is the mean of the initial and final forces. If K is the force which produces unit deformation in a spring and s_1 is the initial deformation, the initial force is Ks_1. If s_2 is the final deformation, the final force is Ks_2. The average force for the displacement from s_1 to s_2 is $\frac{K(s_1 + s_2)}{2}$ —

$$U = \frac{K(s_2 + s_1)(s_2 - s_1)}{2} = \frac{K(s_2^2 - s_1^2)}{2} \qquad (1)$$

Equation (1) gives the increase of potential energy.

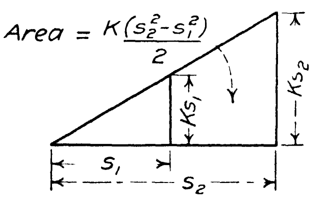

FIG. 329.

Work is often represented by the area of a diagram in which force is the ordinate and deformation or displacement is the abscissa. Figure 329 is the work diagram for a spring. The diagram is a trapezoid. The parallel sides are Ks_1 and Ks_2, and the distance between the sides is $s_2 - s_1$. If the initial displacement is zero, the diagram is a triangle and the total work is $Ks^2/2$.

Problems

1. In Prob. 1 of Art. 174, what is the work done in deflecting the board 6 in? Solve without the formula.
2. In Prob. 1 of Art 174, what is the work done in deflecting the board 1 ft? What is the work of the last 6 in of the foot? Construct the work diagram to the scale of 1 in. = 50 lb and 1 in. = 0.25 ft.

176. Velocity Produced by Elastic Force. When the force is proportional to the displacement, as in the case of a spring, the expression for the force is

$$P = -Kx \qquad (1)$$

in which P is the force exerted by the spring, x is the displacement, and K is the force when the displacement is unity. The sign is negative because the direction of the force *exerted by the spring* is opposite to the direction of the displacement. If a body of mass m is attached to the spring, as in Fig. 330, the acceleration of this body is given by Formula XXV,

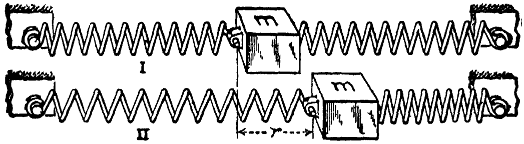

FIG. 330.

$$-Kx = ma \tag{2}$$

Since acceleration along the X axis is d^2x/dt^2, Eq. (2) may be written

$$-Kx = m\frac{d^2x}{dt^2} \tag{3}$$

A differential equation of the form $d^2x/dt^2 =$ *a function of* x is solved by first multiplying both sides by dx.

$$-Kx\,dx = m\,dx\frac{d^2x}{dt^2} = mn\frac{dx}{dt}\frac{d^2x}{dt} \tag{4}$$

Since $dx/dt = v$, $d^2x/dt = dv$, and Eq. (4) may be written

$$-Kx\,dx = mv\,dv \tag{5}$$

From the integration of Eq. (5),

$$-\frac{Kx^2}{2} = \frac{mv^2}{2} + C_1 \tag{6}$$

in which C_1 is an integration constant.

In Fig. 330,I, the mass m is in its position of equilibrium with equal tension on each of the two springs. In Fig. 330,II, the body has been displaced a distance r toward the right and is assumed to be stationary. If let free at this point, the body will move to the position of equilibrium of Fig. 330,I and will continue to move to a distance r on the left of that position. When $x = r$, $v = 0$. Substitution of these values of x and v in Eq. (6) gives

$$C_1 = -\frac{Kr^2}{2} \tag{7}$$

When this value of C_1 is substituted in Eq. (6), the final equation is

$$\frac{mv^2}{2} = \frac{Kr^2}{2} - \frac{Kx^2}{2} \tag{8}$$

Equation (8) was derived from the equation Force = ma. If the right-hand member of the equation represents energy in foot-pounds, the mass in the left-hand member must be in geepounds. The equation is usually written with the mass in pounds and becomes

$$\frac{Wv^2}{2g} = \frac{K}{2}(r^2 - x^2) \qquad \text{Formula XLIV}$$

At the position of maximum deflection, at which $x = r$ and the velocity is zero, the body is at positive *elongation*. The distance r is called the *amplitude*.

Formula XLIV could have been derived directly from the relation of the potential to the kinetic energy, without the use of integration. The integration method here given is general, however, and may be applied to any problem in which the effective force is a function of the displacement. It is advisable, therefore, to use the method in this case where it may be checked in order that it may be employed with confidence in other cases where the physical relations are not so obvious. In numerical problems involving velocity produced by the action of simple linear springs, it is better for the student not to use the formula but to work out the energy relations directly.

Example

A spring suspended from a fixed support is elongated 4 ft when a mass of 24 lb is hung on it. The spring is stretched an additional 3 ft and then released. What will be the maximum velocity of the body? What will be the velocity when the body is 1 ft from the position of equilibrium? What will be the velocity when the body is 2 ft from the position of equilibrium?

This may be solved by means of the energy relations without direct use of Formula XLIV. The maximum velocity is at the position of zero potential energy, which is at the position of equilibrium. To stretch the spring 3 ft requires a pull of 18 lb. The average pull is 9 lb and the displacement is 3 ft. Potential energy at elongation of 3 ft is $9 \times 3 = 27$ ft-lb. All this energy is transferred into kinetic energy as the body returns to the equilibrium position.

$$\frac{24v^2}{2g} = 27 \qquad v^2 = \frac{9}{4} \times 32.174$$

$$v = \frac{3}{2}\sqrt{32.174} = 8.508 \text{ ft per sec}$$

In coming back from 3 ft to 1 ft, the average force is 12 lb and the work done on the body by the spring is 12 × 2 = 24 ft-lb.

$$\frac{24v^2}{2g} = 24$$

$$v = \sqrt{64.348} = 8.022 \text{ ft per sec}$$

In returning 1 ft from the maximum displacement,

$$\text{Work} = \frac{18 + 12}{2} \times 1 = 15 \text{ ft-lb}$$

$$v = 6.342 \text{ ft. per sec}$$

Problems

1. Solve the example above by means of Formula XLIV.
2. Solve the example above if the load of 24 lb stretches the spring 3 ft instead of 4 ft. *Ans.* 9.824 ft per sec; 9.263 ft per sec; 7.323 ft per sec.
3. A mass of 20 lb rests on a smooth horizontal plane attached to a pair of springs as in Fig. 330. The body is at equilibrium at a point A on the plane. It is displaced 4 ft from A by a force of 32 lb and then released. Find the maximum velocity. Find the velocity when it is 0.5 ft from A. *Ans.* 14.35 ft per sec; 14.24 ft per sec.
4. Solve Prob. 3 if the coefficient of friction between the mass and the plane is 0.2. *Ans.* Maximum velocity = 12.56 ft per sec.
5. In Prob. 4, what is the velocity when the mass has returned to A from a point 4 ft to the right of A? *Ans.* 12.43 ft per sec.
6. In Prob. 5, how far will the body swing to the left of A? How far will it swing to the right of A on the return? How far to the left of A the next time? How far to the right the next time? *Ans.* First time 3 ft to the left.
7. In Prob. 6, how much potential energy was stored in the spring when the body was stationary with a displacement of 4 ft toward the right? When the body finally comes to rest account numerically for this energy.

177. Vibration from Elastic Force. From Eq. (8) of Art. 176, which applies to the force of a spring or to any force which is proportional to the displacement and is opposite to the direction of the displacement, the velocity is

$$v^2 = \frac{K}{m}(r^2 - x^2) \tag{1}$$

$$\frac{dx}{dt} = \sqrt{\frac{K}{m}} \times \sqrt{r^2 - x^2} \tag{2}$$

$$\frac{dx}{\sqrt{r^2 - x^2}} = \sqrt{\frac{K}{m}}\, dt \tag{3}$$

$$\sin^{-1}\frac{x}{r} = \sqrt{\frac{K}{m}}\, t + C_2 \tag{4}$$

If time is measured from positive elongation, where $x = r$, then

$$C_2 = \sin^{-1} 1 \qquad \sin C_2 = 1$$

$$C_2 = \frac{\pi}{2} \tag{5}$$

$$\sqrt{\frac{K}{m}}\, t = \sin^{-1}\frac{x}{r} - \frac{\pi}{2} = \cos^{-1}\frac{x}{r} \tag{6}$$

$$\cos\sqrt{\frac{K}{m}}\, t = \frac{x}{r} \tag{7}$$

$$x = r\cos\sqrt{\frac{K}{m}}\, t \tag{8}$$

Equation (8) gives the displacement of the mass in terms of the time which has elapsed since it was at positive elongation. If time were measured from the position of zero displacement, the equation would be

$$x = r\sin\sqrt{\frac{K}{m}}\, t \tag{9}$$

If K in Eq. (8) is expressed in pounds force, then m must be given in slugs. If W is the mass in pounds,

$$m = \frac{W}{32\ 174}$$

Equation (8) gives the displacement in terms of the mass, the constant of the spring, and the interval which has elapsed since the body was at positive elongation. When $x = r$, $t = 0$; when $x = 0$,

$$\cos\sqrt{\frac{K}{m}}\, t = 0$$

hence when $x = 0$,

$$\sqrt{\frac{K}{m}}\, t = \frac{\pi}{2} \tag{10}$$

$$t = \frac{\pi}{2}\sqrt{\frac{m}{K}} = \frac{\pi}{2}\sqrt{\frac{W}{Kg}} \tag{11}$$

Equation (11) gives the time from positive elongation to zero displacement, which is one-half the time of a single vibration. The time of a single vibration, from positive elongation to negative elongation, or from zero displacement to positive elongation and back again to zero displacement, is given by

$$t = \pi\sqrt{\frac{m}{K}} = \pi\sqrt{\frac{W}{Kg}} \tag{12}$$

The time of a *complete period*, from any point to the same point in the same direction, is

$$t_c = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{W}{Kg}} \tag{13}$$

The number of complete periods executed in a unit of time is called the *frequency* of the vibration, which is

$$f = \frac{\sqrt{Kg}}{2\pi\sqrt{W}} \tag{14}$$

Equations (8) and (9) are identical with the coordinates of point B in the circle of Fig. 324 if the angular velocity ω of the line OB is replaced by $\sqrt{K/m}$. The expression $\sqrt{K/m}$ or $\sqrt{Kg/W}$ is called the *circular frequency* of the vibration and is sometimes written ω_n or ω_c, the subscript referring to the *natural* or *characteristic* frequency of the vibration. The type of vibration discussed above is called *free vibration*, as it represents the motion of a mass actuated by a spring without other varying forces, to distinguish it from *forced vibrations* produced by external periodic forces which set up vibrations at frequencies other than the natural ones.

The whole of the analysis above has served simply to prove that the motion of a body actuated by a simple spring, with or without the addition of a constant force, is *simple harmonic motion*, or the projection of uniform circular motion upon a diameter. Every simple harmonic motion can be referred to a circle drawn with the center at the point of maximum velocity and a radius equal to the amplitude of the vibration. Such a *reference circle* is a mnemonic device which can and should be used by the student to determine the times of complete or partial cycles of harmonic motion.

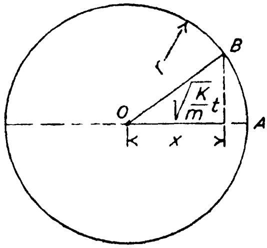

FIG 331

Figure 331 represents a circle of radius r. The angle between the horizontal diameter and the radius OB is $\sqrt{K/m}\,t$ radians. The projection of OB on the horizontal diameter is

$$x = r\cos\sqrt{\frac{K}{m}}\,t$$

which is Eq. (8). If the point B moves around the circumference of a circle with a uniform angular velocity of $\sqrt{K/m}$ radians per second, the projection of B on a diameter of the circle corresponds with the position of a body which vibrates back and forth along this diameter under the action of a force $-Kx$. In Fig. 332, A, B, C, D, E, etc., are equidistant points on the circumference of a circle. If a point moves on this circle with uniform speed, it passes from one of these points to the next one in the same interval of time. The points B', C', D', etc., are the projections of the points B, C, D, etc., on the horizontal diameter of the circle. A body, which is vibrating along the straight line AM, passes from A to B', from B' to C', from C' to D', etc., in equal time intervals.

To use the reference circle, first locate the point of maximum velocity (the center). This is the point where the body has no acceleration.

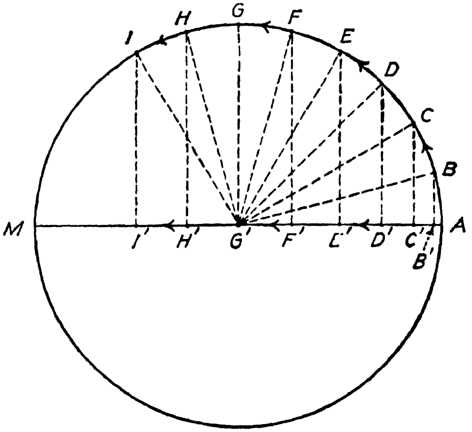

FIG. 332.

At this point the forces acting on the body are in equilibrium. Then find the amplitude of the motion by an energy equation or otherwise. At maximum elongation all the energy of the system is in the spring. The circle can now be drawn. The maximum velocity of the body can be found by an energy equation if there are enough data for the solution of the problem. The maximum velocity is the velocity at which the reference point moves around the reference circle. To time the motion from any point to any other, project the two points on the path onto the reference circle and compute the arc length between the two projected points. This length divided by the maximum velocity gives the time required.

For the time of a complete period, the circumference of the reference circle is divided by the maximum velocity. If the amplitude of the motion is increased, the total energy of the system is increased in proportion to the *square* of the amplitude and the maximum velocity is increased in proportion to the amplitude. The time for a complete period is unchanged. Vibration resulting from a force proportional to the displacement is *isochronous.*

Example

Find the time of a complete vibration of the 20-lb mass of Prob. 3 of Art. 176. Find the position and velocity 0.3 sec after positive elongation. Find the time required to return 3 ft from positive elongation.

The amplitude is 4 ft and the circumference of the reference circle is 25.1328 ft. The maximum velocity was found to be 14.35 ft per sec.

$$t_c = \frac{25.1328}{14.35} = 1.751 \text{ sec}$$

In 0.3 sec the distance traveled on the reference circle is 4.305 ft, which equals 1.07625 rad, or 61°40′.

$$x = 4 \cos 61°40' = 4 \times 0.4746 = 1.898 \text{ ft}$$
$$v = 4 \sin 61°40' = 14.35 \times 0.8801 = 12.63 \text{ ft per sec}$$

At a displacement of 1 ft, the cosine of the angle between OA and OB of Fig. 331 is 0.25. The angle = 75°31′ = 1.3180 rad. Distance on the circumference = 5.2721 ft.

$$\text{Time} = \frac{5.2721}{14.35} = 0.367 \text{ sec}$$

Problems

1. Find the time of a complete vibration of the 24-lb mass of the example of Art. 176. From this answer find the time to return 1.5 ft from positive elongation. *Ans.* 2.215 sec; 0.369 sec.
2. In Prob. 1, how far will the mass rise from the bottom of its path in 0.5 sec? *Ans.* 2.546 ft.
3. Solve Prob. 1 for the time of a complete vibration by Eq. (13).
4. Solve the example above for the time of a complete vibration by Eq. (13).
5. A body sliding on a smooth horizontal plane at 6 ft per sec strikes a spring and is brought to rest in a distance of 4 ft. Find the time to stop. *Ans.* $\pi/3$ sec.
6. If the body of Prob. 5 is a freight car weighing 40 tons and the spring is the bumper sometimes used to stop such cars, find the constant and the maximum force, assuming the bumper to have the characteristics of a simple spring. *Ans.* 11.2 tons; 2.8 tons per ft.

178. Sudden Loads. When a load is applied to an elastic body, the force exerted on the elastic body may be much greater than the weight of the load. Figure 333 represents a helical spring on which is placed a mass of m pounds. Figure 333,I shows the natural length of the spring. In Fig. 333,II, the mass m is attached to the spring but is held up by the support B so that none of its weight comes on the spring. If the support B is lowered slowly until all the weight comes on the spring, the condition of Fig. 333,III is reached. The mass comes to rest when the spring is stretched a distance *d*. If *K* pounds is sufficient to stretch the spring 1 foot,

$$d = \frac{m}{K} \tag{1}$$

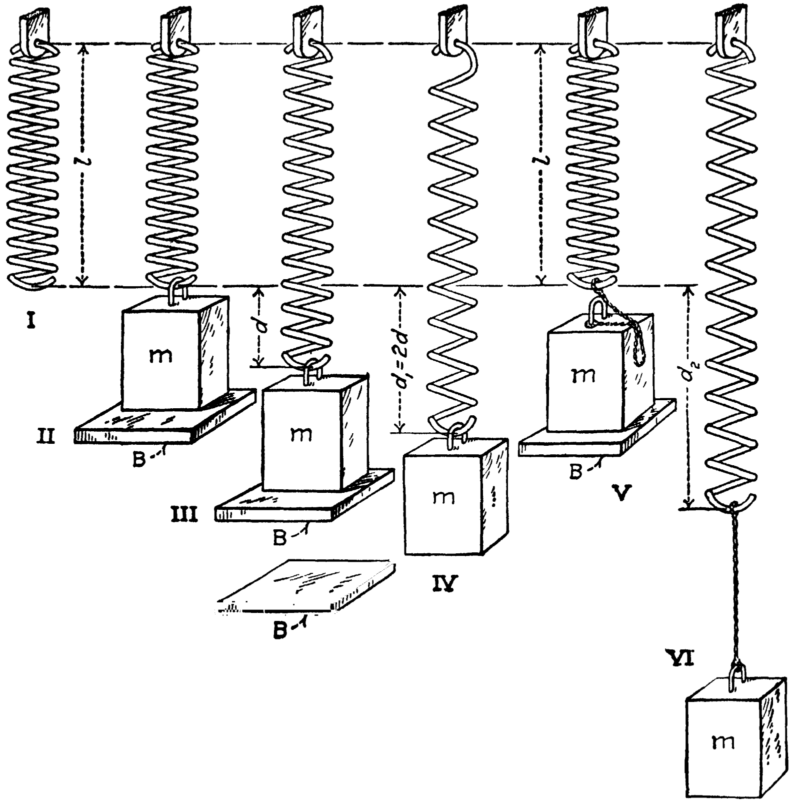

Fig. 333.

The work done by gravity on the mass *m* while it moves a distance of *d* feet is

$$U = md = \frac{m^2}{K} \tag{2}$$

The work done by the mass on the spring, or the potential energy acquired by the spring, is

$$U = \frac{0 + m}{2} d = \frac{md}{2} = \frac{m^2}{2K} \tag{3}$$

Equation (3) represents only half the work of Eq. (2). It is necessary to account for the remainder. At the beginning of the descent, the entire weight of the body rested on the support *B*. As the body was lowered, the spring took an increasingly greater proportion until it finally carried the entire load and the reaction of the support became zero. The work done on the support *B* was

$$\frac{m + 0}{2} d = \frac{md}{2} \tag{4}$$

Equation (4) accounts for the remaining half of the energy.

If the support *B* is suddenly removed when there is no tension in the spring, the motion is that shown in Fig. 333,IV. No work is done on the support and all the energy is stored in the spring. If d_1 is the total elongation of the spring, the average tension is $Kd_1/2$ and the potential energy of the spring is $Kd_1^2/2$. The work of gravity on the mass is md_1. Equating the work and energy,

$$md_1 = \frac{Kd_1^2}{2} \tag{5}$$

$$d_1 = \frac{2m}{K} \tag{6}$$

A comparison with Eq. (1) shows that the elongation when the load is "suddenly applied" is twice as great as when the load is gradually applied. When a load is suddenly applied to a spring or other elastic body in which the stress is proportional to the deformation, the deformation is twice as great as that due to the weight of the load. Since the force is proportional to the deformation, it follows that the force is likewise twice as great as the weight of the load. This fact is expressed by the statement "Impact load is twice the static load."

Since the tension in the spring at the bottom of its path in Fig. 333,IV is twice the weight of the body, the resultant force *upward* in that position is *m* pounds. The body will move upward to the position of Fig. 333,II and will continue to vibrate for a long time. It will finally come to rest in the position of Fig. 333,III.

In Fig. 333,V, the body is suspended from the spring by an inelastic cord which permits it to fall a distance *h* before it brings any tension on the spring. If d_2 is the distance which the spring is stretched, the work of gravity on the mass is $m(h + d_2)$, and the work done on the spring is $Kd_2^2/2$.

$$m(h + d_2) = \frac{Kd_2^2}{2} \tag{7}$$

When a load falls upon an elastic body, the deformation is greater than twice the deformation caused by the weight of the load. The body will vibrate according to the laws of a simple harmonic motion from the position of no deformation of the elastic body down to the lowest point and back again. After it reaches the position at which there is no deformation of the elastic body, the load will continue to move as a projectile which is thrown straight upward. After reaching the starting point, it will fall and repeat the cycle.

Problems

1. A force of 10 lb stretches a given spring 6 in. How much is this spring stretched when a mass of 40 lb is hung on it so that the load is gradually applied?
2. How much is the spring of Prob. 1 stretched if the mass of 40 lb is put on suddenly? What is the tension at the lowest point? What is the time required to reach the lowest point?
3. The 40-lb mass of Prob. 1 falls on the spring from a height of 1 ft. How much is the spring compressed? *Ans.* 4.828 ft.
4. In Prob. 3, what is the entire time required for the mass to fall, compress the spring, and return to its original position?

At the position of equilibrium, the spring is compressed 2 ft. The amplitude of the vibration below this point is 2.828 ft. Consequently, the radius of the circle of reference is 2.828 ft. The angle θ of Fig. 334 is given by the equation

$$\cos\theta = \frac{2}{2.828}$$
$$\theta = 45^\circ$$

If the entire vibration were a simple harmonic motion, the time of the complete period would be $t_c = 1.566$ sec. The time required to traverse 270° of the reference circle is 1.174 sec. The time required to fall 1 ft is 0.249 sec. The entire time is $1.174 + 0.498 = 1.672$ sec.

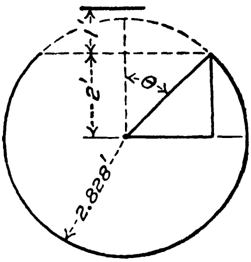

FIG. 334.

5. A 40-lb body starts from rest and slides down a 30° inclined plane against a friction force of 10 lb. After sliding 6 ft it strikes a spring designed to stop the body. If the spring has a constant of 20 lb per ft and itself has a negligible mass, find the time and distance to stop after contact with the spring.
   Ans. Body stops in 3 ft; maximum velocity is $\sqrt{3.125g}$ when the spring is compressed 0.5 ft; time $= 1.41\pi/\sqrt{3.125g}$.
6. Find the distance the body of Prob. 5 will return up the plane and the time for the return. *Ans.* 3 ft; $\pi\sqrt{2/g}$ sec.
7. Where will the body of Prob. 5 finally come to rest?
   Ans. 7 ft down from its original starting point at the end of its second downward movement.
8. Check Prob. 7 by equating the change of potential energy to the total loss to friction plus the final potential energy of the spring.
9. A 10-lb mass is at rest on top of a spring whose constant is 20 lb per ft when a second 10-lb mass dropped from above strikes the first mass with a velocity of

8 ft per sec and adheres. Find how far the combined mass will go down before stopping and the time to arrive.
Ans. Center of reference circle is 0.5 ft below initial position of first 10-lb mass; radius of reference circle is 0.863 ft; arc of circle for required time is 90° + 35°27′; $t = 0.794$ sec.

179. Damped Vibrations. Free vibrations do not persist forever because energy is dissipated by the air resistance and by an effect called *elastic hysteresis* in the springs which participate in the vibration. There may also be ordinary friction forces exerted by surfaces along which the vibrating body slides. As energy is gradually lost from the system, the maximum velocity of the motion decreases. The amplitude also decreases but not always in the same ratio, so that the time of the complete cycle with some types of damping is not exactly the same as the time for the ideal frictionless system. Problems 4 to 6 of Art. 176 and Probs. 5 to 7 of Art. 178 dealt with vibrations resisted by constant friction forces. With this kind of friction, the time of vibration is the same as for the same system without friction. Suppose such a vibration to be initiated by moving the body some distance x away from the position where the spring force is zero. As the body moves back from this position, the restoring force is $Kx - F$, where F is the force of friction. The basic equation of the motion is then

$$-(Kx - F) = m\frac{d^2x}{dt^2} \tag{1}$$

The body is in equilibrium at the point where the spring force equals the friction force, or a distance F/K from the point where the spring force is zero. In Eq. (1), let $x = x' + (E/K)$ and Eq. (1) becomes

$$-Kx' = m\frac{d^2x'}{dt^2} \tag{2}$$

Equation (2) is identical with Eq. (3) of Art. 175 except that x is replaced by x'. The two equations are identical since the x of the one and the x' of the other are the distance of the body from the *position of equilibrium.* The total distance the body will travel during the first vibration is $2x'$ and this motion can be followed by a point moving uniformly along a semicircle of radius x' at a velocity equal to the maximum velocity of the body during this vibration. But for the next vibration, the position of maximum velocity will be at a distance F/K *on the opposite side* of the position of zero spring force. The center of the next reference semicircle is at this new position and the radius is

$x' - 2(F/K)$. The *time* of any vibration is the same as that of the first. The succession of reference semicircles forms the pseudospiral shown in Fig. 335,I. Figure 335,II shows the displacement of the body from the position of zero spring force plotted against time in units of one-quarter cycle. The motion is not difficult to visualize or to calculate exactly, but there is no simple mathematical expression for the displacement as a function of time. The motion ends abruptly the first time the body stops anywhere between the two centers of maximum velocity, since in this region the force of the spring is not sufficient to overcome the friction force.

If the damping force is proportional to the velocity of the body instead of constant, it can be shown that the reference figure is a true spiral and that, ideally, the motion would never cease. In this case there is a simple mathematical function to express the displacement-time relation. This kind of damping is called *viscous damping,* and it represents the effect of internal hysteresis nearly enough to make the equation for the motion of practical use in engineering. If the resistance constant, or the resisting force at unit velocity, is represented by c, the equation of vibration with viscous damping is

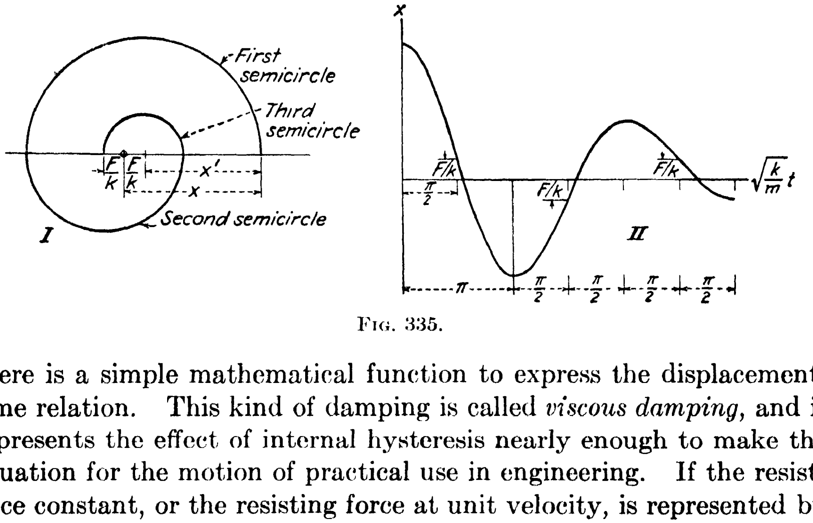

FIG. 335.

$$x = Ae^{-ct/2m} \cos \frac{\sqrt{4mK - c^2}}{2m} \tag{3}$$

if time is measured from the instant when the body was at a distance A from the position of equilibrium and at positive elongation. In this expression K is the spring constant, m is the mass of the body (in proper units), and t is the time. The appearance of $-c^2$ under the radical sign indicates that the total time of a cycle with this kind of damping is a little greater than that for an undamped system with the same mass and spring.

Problems

1. A mass of 50 lb is attached to two horizontal springs as in Fig. 330 and placed on a horizontal table where the coefficient of friction is 0.2. The constant of each of the springs is 20 lb per ft and there is enough initial tension so that the springs do not buckle and get out of line during the motion. If the body is displaced 4 ft from the position where the two spring forces are equal and there released, where will the body finally come to rest and what time will elapse before this happens?

   Ans. Stops where spring forces are equal after $4\pi\sqrt{5/g}$ sec.
2. Check Prob. 1 by comparing the original potential energy in the springs with the friction force multiplied by the total distance of sliding.
3. A body hanging from a spring runs up and down between vertical guides. If the region in which the body will remain at rest is 0.5 in. long, how many vibrations will the body make before coming to rest when it is released at a point 2 in. from the middle of the region of rest?

180. Forced Vibrations. A mass which is free to move, even though its motion is impeded by friction or attachments like springs, can, obviously, be forced to move in any desired manner if suitable forces are applied to it. If the motion produced by the applied forces is a periodic motion, it is called a *forced vibration*. When the forced vibration is to be a simple harmonic vibration, it is not difficult to determine what kind of force is necessary, provided the motion of the mass is impeded only by a simple, linear spring.

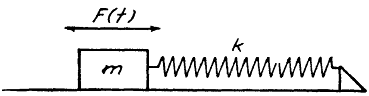

FIG. 336.

Figure 336 shows a mass m, on a smooth horizontal table, attached to a spring of constant K. It is desired to move the mass in the direction of the spring with a simple harmonic motion of amplitude R and an arbitrary circular frequency of ω. The double-ended arrow above the mass marked $F(t)$ symbolizes some kind of back-and-forth force which varies in some as yet unknown way with time and which is to be applied to the mass to produce the motion

$$x = R \cos \omega t \tag{1}$$

The velocity of the required motion at any time is found by differentiating Eq. (1).

$$\frac{dx}{dt} = -R\omega \sin \omega t \tag{2}$$

The acceleration is found by differentiating again, and

$$\frac{d^2x}{dt^2} = -R\omega^2 \cos \omega t \tag{3}$$

The effective force acting on the mass at time t is

$$F(t) - Kx = F(t) - KR \cos \omega t \tag{4}$$

Putting the expression for the force from Eq. (4) and for the acceleration from Eq. (3) into the equation Force = mass × acceleration gives

$$F(t) - KR \cos \omega t = -mR\omega^2 \cos \omega t \tag{5}$$

It cannot be determined from this analysis whether any other expression for $F(t)$ would do, but it is obvious that if $F(t) = F \cos \omega t$, a simple relation between F and the other elements of Eq. (5) is obtained. Making this substitution for $F(t)$ and canceling $\cos \omega t$, there results

$$F = KR - mR\omega^2 \tag{6}$$

Solving Eq. (6) for R and substituting in Eq. (1)' gives

$$x = \frac{F \cos \omega t}{K - m\omega^2} \tag{7}$$

The force $F \cos \omega t$ is called a *sinusoidal force*, and such a force applied to simple spring-mass systems produces simple harmonic or *sinusoidal motions*. This type of force and consequent motion arises from rotating machinery, vibrating bodies, and electromagnets carrying alternating currents. Equation (7) is a formula for finding the amplitudes resulting from such forces or the required maximum intensity of such forces to produce vibrations of specified amplitudes. The equation can be made to appear simpler by letting $F/K = x_s$, the static deflection or distance the maximum value of the force would extend the spring, and letting $M/K = \omega_c$, the characteristic circular frequency of the free vibration of the mass on the spring without outside forces. With these substitutions, Eq. (7) becomes

$$x = \frac{x_s \cos \omega t}{1 - (\omega/\omega_c)^2} \qquad \text{Formula XLV}$$

When the frequency of the applied force coincides with the natural frequency of the system to which it is applied, the denominator in Eq. (7) or in Formula XLV becomes zero, which seems to imply that the amplitude of the motion would be infinity. A more common-sense interpretation would be that a vibration of any desired amplitude at this frequency, once started, can be maintained without any outside force. Actually, no system can be devised without some damping effects and to keep the amplitude constant a small amount of energy must be put into the system to make up the damping losses. The condition at which the denominator of Formula XLV is zero is called *resonance,* and in this condition a very large amplitude can be maintained with an extremely small sinusoidal force.

The factor $\frac{1}{1 - (\omega/\omega_c)^2}$ of Formula XLV is called the *amplifying factor* since it is a factor by which the static deflection (which would result from a simple application to the system of the maximum force) must be multiplied to predict the amplitude of the motion resulting from the sinusoidal force. Figure 337 shows the amplifying factor plotted against the frequency ratio. This curve is known as the *resonance curve.* Beyond the point where the frequency ratio is unity, the maximum displacement due to the sinusoidal force is in the *opposite direction from the force.* This part of the curve is plotted, for convenience, with the sign reversed. The solid curve extends to infinity at resonance. The other curve shows what would actually happen if the spring-mass system were damped so that each free vibration would have an amplitude 80 per cent as great as the preceding one. The equation for the dotted curve is somewhat more complicated than that for the solid one.

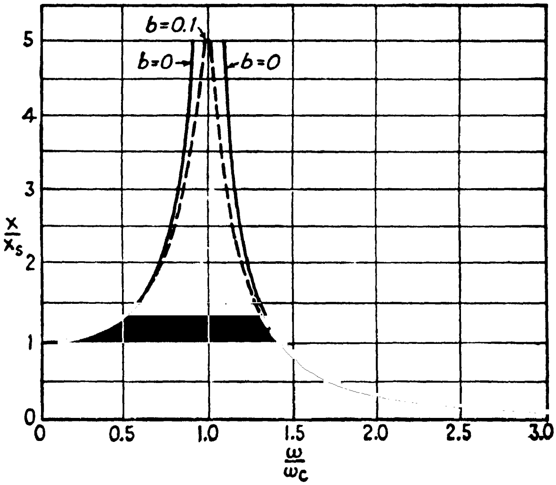

FIG. 337.

In order that the student may have a better understanding of Formula XLV, he should do the two following exercises:

1. Repeat the derivation, assuming instead of Eq. (1) the desired motion to be $x = R \sin \omega t$.

2. Assume that the external force is to be applied to the mass of Fig. 336 by an additional spring, and draw a figure like Fig. 336 with the additional spring between the mass and a support at the left. (*a*) Let the constant of this additional spring be K' and determine (in terms of m, K, and ω) the value of K' needed to cause the mass to execute free vibrations at the circular frequency ω under the combined action of the two springs. (*b*) Find the amplitude R of this vibration to cause the maximum force exerted by the additional spring to be F. (*c*) Substitute this value of R in the equation $x = R \cos \omega t$ or $x = R \sin \omega t$.

Problems

1. A mass of 8.04 lb, mounted on a spring with a constant of 9 lb per ft, has within it a small rotor turning at 120 rpm. If the eccentricity of this motor sets up a disturbing force of 1 oz, what will be the magnitude of the resulting vibration? *Ans.* 0.92 in.
2. Find what spring constant would be needed in Prob. 1 to reduce the vibration to 0.002 in.

181. Torsional Vibrations. In many practical engineering cases bodies are restrained in their angular motions by shafts, rods, springs, or other elastic restraints tending to restore them to their positions of equilibrium with torques which are proportional to their angular displacements. All that has been said about linear vibrations is properly applicable to such torsional systems and all equations may be translated into torsional equations by substituting moments of inertia for mass and angular velocities and accelerations for their linear counterparts. If K is the torque required to displace the body 1 radian from its position of equilibrium and the body on the elastic rod or spring is twisted through an angle β from the position of equilibrium, the work done is $K\beta^2/2$. This work is stored up as elastic energy. If the body is released and permitted to rotate back to the position at which its displacement from the position of equilibrium is θ, the change in elastic energy is the first member of the equation

$$\frac{K\beta^2}{2} - \frac{K\theta^2}{2} = \frac{I\omega^2}{2} \qquad \text{Formula XLVI}$$

The second member of Formula XLVI is the kinetic energy of rotation. If the torque K is expressed in pound-feet, the moment of inertia I is the mass in slugs multiplied by the square of the radius of gyration in feet.

$$I = \frac{W}{g} k^2$$

Formula XLVI for angular motion corresponds with Formula XLIV for linear motion. The formula states that the sum of the potential and kinetic energy is constant and is equal to the potential energy at the maximum displacement.

Problems

1. A homogeneous solid cylinder 1 ft in diameter and weighing 20 lb is suspended with its axis vertical by means of a ¼-in. steel rod 10 ft in length (Fig. 338). A force of 3 lb at the end of an arm 1 ft long is capable of twisting this rod through 1 rad. The cylinder is rotated through an angle of 180° and then released. Find its maximum angular velocity.

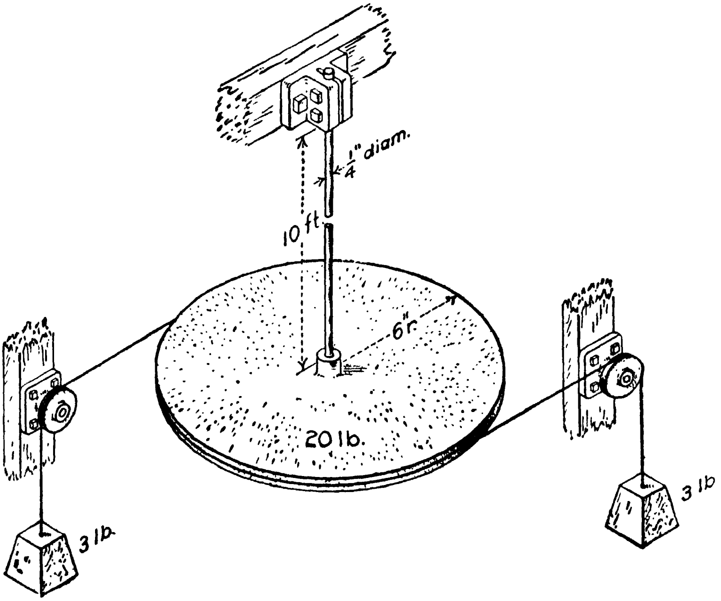

FIG. 338.

$$\textit{Ans. } K = 3 \qquad I = \frac{20}{32.174} \times \frac{1}{4} \times \frac{1}{2} = \frac{2.5}{32.174} \qquad \beta = \pi;\ \theta = 0$$

$$\omega^2 = 3 \times 9.87 \div \frac{2.5}{32.174} \qquad \omega = 19.52 \text{ rad per sec}$$

2. Solve Prob. 1 if the initial displacement is 90°.
Ans. Maximum ω = 9.76 rad per sec.
3. Solve Prob. 1 for the velocity after the cylinder has returned 135° from the maximum displacement. *Ans.* ω = 18.90 rad per sec.
4. Find the time of a complete period of the system shown in Fig. 338 with the two 3-lb masses included in the system. Solve by assuming the masses pulled

down 1 ft and released. Then find the velocity of one of the masses on return and divide this velocity into 2π, the circumference of the reference circle.

182. Period of Oscillation. Formula XLVI is mathematically identical with Formula XLIV for the linear velocity caused by a force which varies as the displacement.

If the circular motion of a point which moves with the angular velocity of Formula XLVI is developed into linear motion, as shown in Fig. 339, this developed motion is simple harmonic. If a sheet of paper is placed on a cylinder which has this motion and a pencil is moved with constant speed parallel to the axis, the pencil mark is the sine curve of Fig. 340.

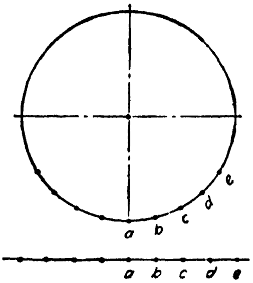

FIG. 339.

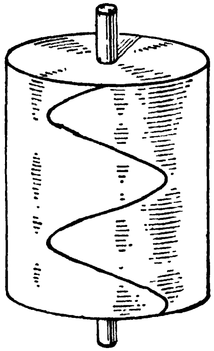

FIG. 340.

Since the angular motion of a body subjected to torque which varies as the angular displacement is a simple harmonic motion, it may be studied by means of a circle of reference. The radius of this circle of reference is the angular amplitude β expressed in radians (Fig. 341). A point may be considered as moving in the circumference of this circle with a linear velocity which is equal to the maximum angular velocity of the vibrating mass. The time of a complete period is the time required for this point to make one complete revolution around the circle of reference. The angular velocity at any instant is the component along the diameter of the circle of this maximum velocity in its circumference.

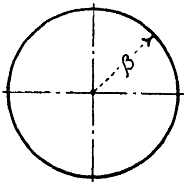

FIG. 341.

The time of a complete linear vibration is

$$t_c = 2\pi \sqrt{\frac{m}{K}} \tag{1}$$

in which m is the mass in slugs and K is the force required to cause unit displacement. For angular vibration,

$$t_c = 2\pi\sqrt{\frac{I}{K}} \tag{2}$$

in which K is the torque which will cause unit angular displacement and I is the moment of inertia. If the torque is in pound-feet, the moment of inertia is the product of the mass in slugs multiplied by the square of the radius of gyration in feet. Equation (2) may be written

$$t_c = 2\pi\sqrt{\frac{Wk^2}{Kg}} = 2\pi\sqrt{\frac{Wk^2}{32.174K}} \tag{3}$$

in which W is the mass in pounds and k is the radius of gyration in feet.

Problems

1. Find the time of vibration of the cylinder of Prob. 1 of Art. 181 by means of the reference circle. Check by Eq. (3). The radius of the reference circle is the amplitude in radians which is π. The circumference is $2\pi^2$.
   Ans. $t_c = \dfrac{2 \times 9.87}{19.52} = 1.011.$
2. A homogeneous solid cylinder is 2 ft in diameter and weighs 64.348 lb. It is suspended with its axis vertical by means of a steel rod and makes 100 complete vibrations in 240 sec. What torque in pound-feet will twist this rod 1 rev?
   Ans. $K = 6.854$ lb-ft; $T = 43.06$ lb-ft.

183. The Gravity Pendulum. A body which is free to rotate through a small angle about a horizontal axis under the action of its own weight is a pendulum. The axis O of Fig. 342 is called the *axis of suspension*. If m is the mass of the pendulum in slugs, the weight in pounds is mg. If $\bar{r}$ is the distance of the center of gravity below the axis of suspension, the work required to displace the pendulum a small angle θ from its position of equilibrium is $mg\bar{r}\theta^2/2$ (see Art. 73), which can be equated to $I_O\omega^2/2$, where ω is the maximum angular velocity of the pendulum on its swing back through the equilibrium position and I_O is the moment of inertia in pounds feet squared. The maximum displacement of any point distant l from 0 (along an arc) is $l\theta$ and the circumference of the reference circle for this point is $2\pi l\theta$. The maximum velocity of this point during the swing is ωl. From the energy equation, $\omega = \theta\sqrt{mg\bar{r}/I_O}$. Dividing the circum-

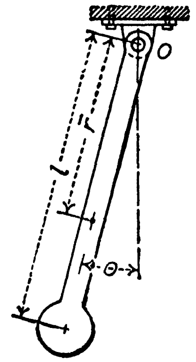

FIG. 342.

ference of the reference circle by the maximum velocity, the time for a complete period is seen to be

$$t_c = 2\pi \sqrt{\frac{I_O}{mg\bar{r}}} \tag{1}$$

Since $I = mk^2$,

$$t_c = 2\pi \sqrt{\frac{k^2}{g\bar{r}}} = 1.1077 \sqrt{\frac{k^2}{\bar{r}}} \tag{2}$$

in which k is the radius of gyration with respect to the axis of suspension and $\bar{r}$ is the distance from the axis of suspension to the center of gravity.

The ideal simple pendulum consists of a point mass suspended from a weightless cord. In this case the length of the cord l is identical with k and $\bar{r}$, so that Eq. (2) becomes

$$t_c = 2\pi \sqrt{\frac{l}{g}} \tag{3}$$

It can be seen from this equation that a simple pendulum extending from the same point of suspension down to the center of percussion of a physical pendulum will have the same period of vibration.

Problems

1. A uniform rod 6 ft long rotates about an axis through one end perpendicular to its length. Find the time of one complete vibration.

   Ans. $k^2 = 6^2/3 = 12$; $\bar{r} = 3$; $t_c = 2\pi \sqrt{4/g} = 2.216$ sec.

2. Find the time of vibration of the rod of Prob. 1 with respect to an axis perpendicular to its length 2 ft from one end. *Ans.* $t_c = 2.216$ sec.

3. A rectangular board of uniform thickness and density is 3 ft long and 1 ft wide. The board vibrates as a pendulum about an axis perpendicular to its plane through the middle of one 1-ft edge. Find the time of vibration.

   Ans. $t_c = 1.589$ sec.

4. Find the time of vibration of a thin rod 20 in. long about an axis perpendicular to its length 4 in. from one end. Use inch units with $g = 386$.

   Ans. $t_c = 1.087$ sec.

5. A thin rod 18 in. long vibrates about an axis perpendicular to its length. Find the value of $k^2/\bar{r}$ when the axis is at the end, 2 in. from the end, 4 in. from the end, 6 in. from the end, and 8 in. from the end.

   Ans. $k^2/\bar{r} = 12$ in., 10.857 in., 10.40 in., 12 in., 28 in.

6. A flywheel is 4 ft in diameter and has its center of mass at the axis. The wheel vibrates as a pendulum about a knife-edge under the rim at 23 in. from the axis (Fig. 343). The wheel makes 100 complete cycles in 196 sec. Find its

radius of gyration with respect to the knife-edge and with respect to the axis through the center of gravity. *Ans.* k = 29.39 in.; 18.30 in.

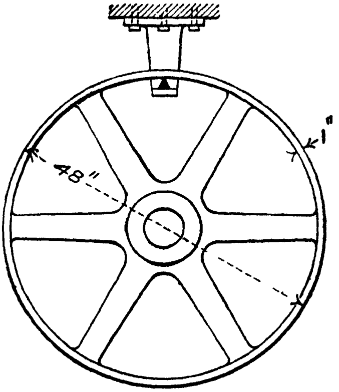

FIG. 343

7. A uniform bar of length l vibrates as a pendulum about an axis perpendicular to its length at a distance of one-third its length from one end. Find the length of the equivalent simple pendulum. *Ans.* $l' = 2l/3$

8. What is the position of the axis of suspension of a uniform bar which gives the minimum time of vibration? What is the length of the equivalent simple pendulum?

$$\textit{Ans.}\ \bar{r} = \frac{l}{\sqrt{12}};\ l' = \frac{l}{\sqrt{3}} = 0.577l$$

CHAPTER 19

MOVING FRAMES OF REFERENCE

184. The Newtonian Frame of Reference. In all that has been said before, it was tacitly assumed that the observer of the bodies under study was at rest in a room or "frame" which itself had no motion. The coordinate axes used to measure displacements and velocities were also at rest relative to this frame. Such a frame of reference is called a *Newtonian frame* after Newton, who conceived a frame of reference absolutely fixed in space with the sun as a center and with no rotation relative to the "fixed" stars. It is within this frame that the simple relation, force is proportional to the product of mass and acceleration, is found to be valid. But since a body under the influence of no force proceeds in a straight line at constant speed, a set of such bodies all with the same velocity would be at rest relative to a frame moving along with them, and the bodies in such a frame would acquire velocities relative to the moving frame in accordance with the basic Newtonian equation. It is for this reason that a frame moving with uniform velocity and without rotation is also called a Newtonian frame.

Probably nowhere will any man ever find himself in a position to make observations in a true Newtonian frame, but any frame fixed relative to the earth is so nearly moving in a straight line with uniform velocity that it is, for most practical purposes, such a frame. There are, however, some practical problems which require that the motion of the earth be taken into account, and there are numerous problems relating to moving, especially rotating, machines and their parts which can best be studied by means of frames of reference fixed relative to some moving part or, in simple words, moving with it. Problems of dynamics and statics in such frames are customarily reduced to equivalent systems in Newtonian frames by the addition of certain fictitious forces acting on the masses under consideration and, in their magnitudes and directions, depending on the motions of the frames as well as the magnitudes of the masses themselves. A proper set of such fictitious forces could be found for a frame moving in any specified manner. Only the frame moving with uniform acceleration and the frame turning with uniform angular velocity will be discussed below, since frames with other types of motion are rarely necessary in engineering work.

185. Equivalent Systems. The system of forces and masses constituting a problem of mechanics can often be replaced by a different (usually simpler) system which will, upon solution, yield some or all of the results obtainable from the original system, the second system being *equivalent* mathematically to the first. Figure 344,I shows the system of Example 2 of Art. 157. The action of this system does not depend on the mass of the 240-pound drum but only on its radius and moment of inertia, so that the drum may be replaced by any other drum of the same radius and moment of inertia as the one in the system. If the homogeneous solid cylinder is replaced by an infinitely thin hollow cylinder of the same radius r and mass m', the moment of inertia will be unchanged if m' is determined so that

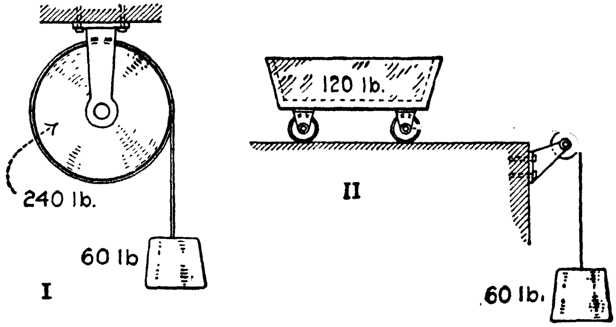

FIG. 344.

$$I = \frac{mr^2}{2} = m'r^2 \qquad m' = \frac{m}{2} \tag{1}$$

in which m' is the *equivalent mass.* Since the linear acceleration of every particle of this imaginary cylinder is the same as that of the rope, it may be replaced by the equivalent mass with linear motion in a horizontal plane as shown in Fig. 344,II. Since the total equivalent mass is 180 pounds and the effective force is 60 pounds, it is evident that the linear acceleration is $g/3$ and that one-third of the gravity force of 60 pounds is required to accelerate the 60 pound mass.

The equivalent mass in this example turns out to be one-half the actual mass, but this is a special case for a homogeneous solid cylinder reduced to equivalent mass *at the periphery.* It may be desired to reduce rotating bodies of other shapes to equivalent masses at any distance r from their centers, in which case

$$m' = \frac{I}{r^2} = \frac{mk^2}{r^2} \tag{2}$$

In the example above and the problems following, no advantage is gained from the use of equivalent systems except that some labor is saved in solving the equations. When there is a large number of connected bodies in the actual system, the time saved may be considerable but the principal value of an equivalent system for the student may be in the better grasp of the physical concepts which its use can give.

Problems

1. A wheel 6 ft in diameter weighs 500 lb. Its radius of gyration is 2.4 ft. What mass concentrated at the rim will have the same moment of inertia? What linear acceleration will be given to the rim by a force of 96 lb applied by a thin belt?
   Ans. Equivalent mass = 320 lb; $a = 0.3g = 9.652$ ft per sec per sec; $\alpha = 3.217$ rad per sec per sec.
2. A wheel 5 ft 11 in. in diameter weighs 600 lb and has a radius of gyration of 2.7 ft. A 1-in. rope hangs over the wheel and carries 40 lb on one end and 240 lb on the other end. The friction is sufficient to keep the rope from sliding. Find the acceleration of the rope, the angular acceleration of the wheel, and the tension in each vertical portion of the rope.
   Ans. $a = 200g/766 = 8.40$ ft per sec per sec; $\alpha = 2.8$ rad per sec per sec.
3. A homogeneous solid cylinder rolls down a 20° inclined plane where the rolling resistance is negligible and the friction is sufficient to prevent sliding. Find the acceleration.
   Ans. The rolling cylinder has 1.5 times the kinetic energy of the same mass in simple translation, and $a = 2g \sin 20°/3$.
4. Assuming that the mass P of Fig. 200 (page 198) is large enough to accelerate downward, convert the system into an equilivalent rotating system and find the angular acceleration.
   Ans. The equivalent moment of inertia equals the moment of inertia of the wheel plus the quantity $(Pr^2 + Wa^2)$. The effective torque is $(Pr - Wa)$.
5. Convert the system of Fig. 201 (with P assumed large enough to cause its acceleration to be downward and the mass of the moving pulley negligible) into an equivalent torsional system and find the angular acceleration. Check by a direct solution by writing separate equations for each of the several bodies.
6. A mass of 10 lb suspended by a cord from a spring is found to vibrate at the rate of 90 complete cycles per min. When the cord between the spring and the 10-lb mass is looped around a pulley 3 ft in diameter mounted to turn freely on a horizontal axis, the frequency is reduced to 10 cycles per min. Find the moment of inertia of the pulley. *Ans.* 45 lb-ft^2.

186. D'Alembert's Principle. If a is the acceleration of a body in any given direction, and P_1, P_2, . . . , are components of the applied forces in that direction, Newton's second law may be written

$$P_1 + P_2 + \cdots = \Sigma P = ma \tag{1}$$

in which ΣP represents the sum of the components of the applied forces in the direction of the acceleration. This is the *effective* force. Equation (1) may be written

$$\Sigma P - ma = 0 \tag{2}$$

In this form the equation represents D'Alembert's principle. The quantity represented by ma is now called the *reversed effective force.* If a body or portion of a body is accelerated by a system of forces, a *fictitious force* which is equal to the product of the mass multiplied by the acceleration and in a direction opposite to the acceleration may be added to the applied forces and the resulting equation will be an equation of equilibrium. When the motion of the body is a pure translation, the reversed effective force is applied at the mass center of the body to reduce the problem to an equivalent problem of statics. When the body has angular as well as linear acceleration, a reversed effective force must be applied to each *particle* of the body. The resultant of the reversed effective forces on the whole body is the equilibrant of the actual external force system acting on the body and in general does not act along a line passing through the mass center. In rotating bodies, the radial component of the reversed effective force is often called *centrifugal force.* D'Alembert's principle is not an essential principle of mechanics. In making use of the principle, the student should avoid logical confusions by keeping in mind that the revised system is an *equivalent system* and that the D'Alembert forces have no reality in fact. The use of the principle of D'Alembert is in reality a consideration of the problem from the standpoint of a moving frame of reference and, as such, it has ample logical justification.

Example

Find the reactions in Fig. 301 by the use of D'Alembert's principle. Assuming that the body will move to the right with a linear acceleration but without any angular motion, the resultant of all forces is 200 lb horizontally to the right through the center of gravity. Add to the figure a fictitious force of 200 lb through the center of gravity and toward the left. Take moments about the point where the rear wheel touches the rail.

$$200 \times 2 + 4R_2 = 200 \times 3 + 800 \times 2 \qquad R_2 = 450 \text{ lb}$$

Problems

1. Finish the example by finding R_1 by moments and check by vertical resolution.
2. An automobile with rear-wheel drive has a 10-ft wheel base and a center of gravity 4 ft above the road and 6 ft forward of the rear axle. Find the maximum forward acceleration as limited by a coefficient of friction of 0.6 at the driving wheels *Ans.* $10g/19$.
3. Three small 1-lb masses are held by three light rods at the angles of an equilateral triangle with sides 2 ft long. Find the tension in the rods due to rotating the system about its mass center at 30 rad per sec. *Ans.* $600/g$ lb.

187. Reference Frame with Linear Acceleration. Figure 345 represents a frame Oxy fixed in space and another frame $O'x'y'$ moving with a uniform acceleration. Point P is moving in any manner relative to $O'x'y'$, and its coordinates with respect to this frame are x', y'. Let the x component of the acceleration of the frame $O'x'y'$ be b and the y component be c. The coordinates of O' with respect to Oxy are u and z. The coordinates of P with respect to Oxy are $(u + x')$ and $(z + y')$. The x components of the velocity of P relative to the fixed frame are

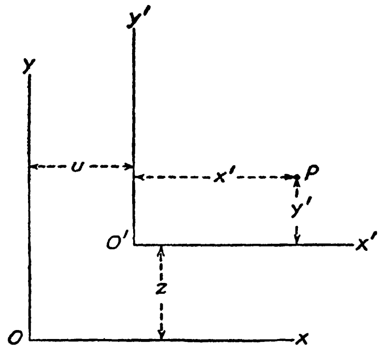

FIG. 345.

$$v_x = \frac{d}{dt}(u + x') = \frac{du}{dt} + \frac{dx'}{dt} \qquad (1)$$

$$v_y = \frac{d}{dt}(z + y') = \frac{dz}{dt} + \frac{dy'}{dt} \qquad (2)$$

and the components of the acceleration of this point are

$$a_x = \frac{dv_x}{dt} = \frac{d^2u}{dt^2} + \frac{d^2x'}{dt^2} = b + \frac{d^2x'}{dt^2} \qquad (3)$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2z}{dt^2} + \frac{d^2y'}{dt^2} = c + \frac{d^2y'}{dt^2} \qquad (4)$$

Equations (3) and (4) show that the acceleration of the point P relative to the fixed frame is simply its acceleration relative to the moving frame added (vectorially) to the acceleration of the moving frame. When the moving frame is moving with uniform velocity, b and c are zero, and the acceleration of the point P relative to the moving frame is also its absolute acceleration. This was stated without proof in Art. 184.

In the process of solving problems in which accelerating frames of reference are used, the acceleration of the frame must be added to the acceleration of each body relative to the frame to get the correct acceleration, or there must be applied to each body under consideration a reversed effective force corresponding to the acceleration of the frame of reference which is being used. The second alternative often seems much simpler. The set of fictitious forces added to the system constitutes a field of force in every way resembling gravitation in a direction opposite to the acceleration of the frame and of magnitude proportional to the masses and to the acceleration of the frame. If the frame is accelerating to the right at $0.25g$, every mass under consideration, whatever its velocity and acceleration, is considered to have acting on it a force of 0.25 of its weight to the left in addition to the actual gravity force, its weight, downward. The result of this modification of the actual system is an equivalent system to which the basic laws of mechanics are strictly applicable. In the special cases where no part of the system under consideration is accelerated relative to the reference frame, the equivalent systems make each problem a problem of statics such as the example and problems of Art. 186.

Example 1

While a train on level track is accelerating forward at $0.1g$, a 60-lb suitcase is moved from the floor at the rear of the car and placed on a rack 40 ft toward the front and 6 ft above the floor. Find the work done.

Work done against gravity	60×6	= 360 ft-lb
Work against D'Alembert's force	6×40	= 240 ft-lb
		600 ft-lb total

Example 2

Find the horizontal and vertical components of the velocity required to throw a ball from the rear of the car of Example 1 to a point at the same elevation and 60 ft forward if the maximum height of the trajectory above the end points is to be 4 ft. Assume $g = 32$ ft per sec per sec.

Let v_y and v_x be the components of the initial velocity relative to a frame of reference moving with the car. Acceleration components are downward 32 ft per sec per sec due to gravity and 3.2 ft per sec per sec horizontally toward the rear of the car due to D'Alembert's force. Let W = the weight of the ball.

$$\frac{Wv_y^2}{2g} = 4W \qquad v_y = 16$$

Average upward velocity $= {}^{16}\!/_{2} = 8$ ft per sec
Total time of flight $= 2 \times {}^{4}\!/_{8} = 1$ sec
Final horizontal velocity $= v_x - 3.2t = v_x - 3.2$
Average horizontal velocity $= v_x - 1.6$

$$(v_x - 1.6)1 = 60 \qquad v_x = 58.4 \text{ ft per sec}$$

Problems

1. Check Example 2 by equating the change in kinetic energy during the flight to the work done against D'Alembert's force.
2. Solve Example 2 if the maximum height of trajectory is 9 ft.
3. Find the angle with the vertical at which a simple pendulum will hang at rest relative to the car of Example 1 and find the time of a complete period of such a pendulum in the car. *Ans.* 5°43′; $2\pi\sqrt{l/1.005g}$.
4. Find the work required to lift a 40-lb body 4 ft from the floor of an elevator accelerating at $0.5g$ upward and replace it while the elevator is accelerating at $0.25g$ downward. *Ans.* 40 ft-lb.
5. Does it make any difference which way the elevator of Prob. 4 is moving during either part of the action of lifting and replacing the body?
6. A 40-lb body hangs from a spring whose constant is 20 lb per ft in an elevator while it accelerates upward at $0.25g$. How much is the spring stretched at the equilibrium and what is the circular frequency of the free vibration about this position? *Ans.* 2.5 ft; $\sqrt{0.5g}$.
7. A 200-lb block of ice 5 ft back of the cab in an ice truck running on a level road moves forward relative to the truck when the brakes are applied. Suppose the ice can move 5 ft before striking the cab of the truck, the coefficient of friction between ice and floor is 0.1, and the brakes are applied to cause an acceleration of $0.25g$. Find the time for the ice to strike the cab and the amount of energy dissipated by the impact. *Ans.* 1.44 sec; 150 ft-lb.
8. A body is given an initial lateral velocity across the floor of a car accelerating horizontally at $0.38g$. The coefficient of friction between the body and the floor is 0.2 and it is required to determine the path of the body. The problem being beyond the power of simple mathematics, it is desired to set up an inclined plane model to obtain the answer experimentally. Find the necessary angle of inclination and coefficient of friction. *Ans.* 22°20′; $\mu = 0.216$.
9. An airplane climbs at an angle of 30° while accelerating at $0.2g$ in the direction of its path. Assume that the floor of the plane is level laterally and parallel to the flight path longitudinally. A body is released on the floor and slides 6 ft to the rear. Find its velocity relative to the plane if the coefficient of friction is 0.125. *Ans.* 15.2 ft per sec.

188. Rotating Reference Frame. In dealing with the motion of bodies in a frame of reference with uniform acceleration, it was necessary only to add to the acceleration of the bodies relative to the frame the acceleration of the frame itself. In the case of a frame with angular velocity, however, every point in the frame has an acceleration toward the center of rotation of the frame, the familiar $r\Omega^2$, and every body moving within the frame has an acceleration, $2v\Omega$, arising from its velocity. To get the absolute acceleration of a body moving in the frame, these two accelerations must be added vectorially to the acceleration of the body relative to the frame. The problem is of considerable practical importance. The formulas for the position and velocity accelerations are relatively simple, and practical problems are greatly simplified by the use of reversed effective forces corresponding to these accelerations. When these forces are added to the system it becomes an equivalent system in a Newtonian frame. Numerous derivations of the two expressions for acceleration given above may be found in works on mechanics. The following one is presented for its simplicity rather than for its elegance.

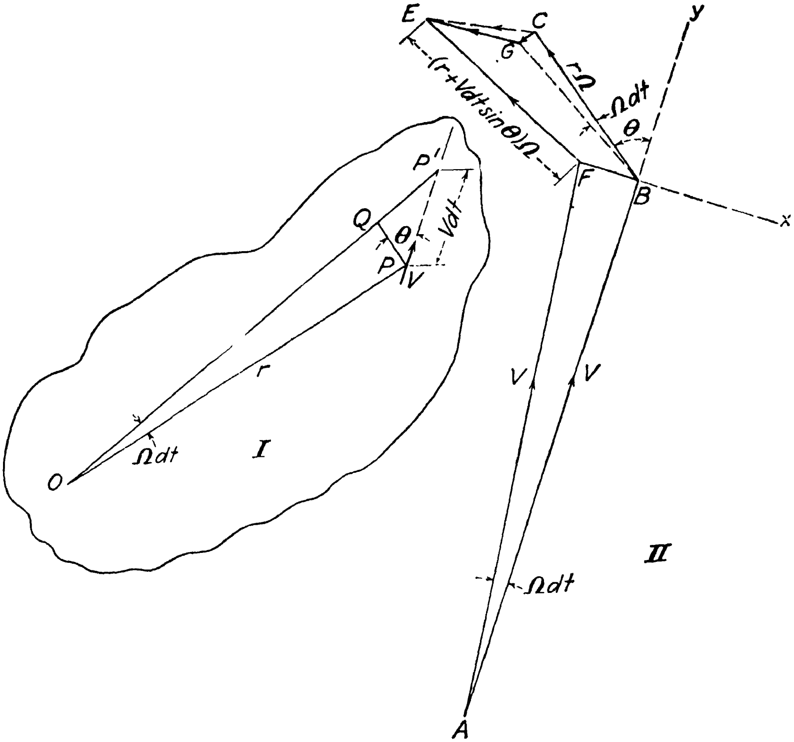

FIG. 346.

In Fig. 346,I, let P and P' be two points fixed in a plane which is rotating counterclockwise about the center O with an angular velocity Ω, and let r be the distance OP. A moving point at P has the velocity v toward P' and travels from P to P' in the short interval of time dt. Let the angle $POP' = \Omega\, dt$ and let $OQ = OP$. Let the angle $QPP' = \theta$. The moving point passes from P to P' along its path in the rotating plane in the same time that the line OP rotates into the absolute position formerly occupied by OQ.

As the moving point passed through P, its absolute velocity was the vector sum of v and $r\Omega$, the absolute velocity of P. These two vectors are shown in Fig. 346,II, AB being v in its initial absolute direction and BC being $r\Omega$ normal to the initial position of OP. When the moving point arrives at P', the direction of PP' will have changed by the angle $\Omega\,dt$ and v will have the direction AF. The absolute velocity of the point P' is

$$(r + QP')\Omega = (r + v\,dt \sin\theta)\Omega$$

In Fig. 346,II, AF and FE are the components of the absolute velocity of the moving point as it passes P', and the vector difference CE is the change in the absolute velocity of the moving point in the time dt.

Let the vector difference CE be resolved into the two components CG and GE, and let CG be made arbitrarily equal to $r\Omega^2\,dt$ in the direction PO. It remains only to show that, when dt is made infinitesimal, GE becomes $2v\Omega\,dt$ and its direction is perpendicular to v. Both components must be divided by dt to arrive at the central and lateral components of acceleration of a point moving with uniform velocity relative to a rotating plane.

Let B be the origin of the rectangular coordinates Bx and By, with By in the initial direction of v.

Since CG was made equal to $r\Omega^2\,dt$, the angle $CBG = \Omega\,dt$. $BF = v\Omega\,dt$ along the x axis when dt approaches zero. FE is normal to OP' *after OP' has rotated through the angle* $\Omega\,dt$ and hence makes the angle $2\Omega\,dt + \theta$ with the y axis. The ordinate of E minus the ordinate of G is

$$y_E - y_G = (r + v\,dt \sin\theta)\Omega\cos(\theta + 2\Omega\,dt) - r\Omega\cos(\theta + \Omega\,dt) \quad (1)$$

Upon expanding the cosines of the sums, replacing cosines of small angles by unity and sines of small angles by the angles themselves, and dropping the term with the factor dt^2, Eq. (1) becomes

$$y_E - y_G = r\Omega^2\,dt\sin\theta - v\Omega\,dt\sin\theta\cos\theta \quad (2)$$

From Fig. 346,I, $v\,dt\cos\theta = PQ = r\Omega\,dt$, and when this substitution is made the right-hand member of Eq. (2) becomes zero. *The component GE is perpendicular to the velocity of the moving point at P.*

Making x positive toward the left, noting that $BF = v\Omega\,dt$, the difference of the abscissas of E and G is

$$x_E - x_G = v\Omega\,dt + \Omega(r + v\,dt\sin\theta)\sin(\theta + 2\Omega\,dt) - r\Omega\sin(\theta + \Omega\,dt) \quad (3)$$

Expanding the sines of the sums and reducing as before, this becomes

$$x_E - x_G = v\Omega\,dt + r\Omega^2\,dt\cos\theta + v\Omega\,dt\sin^2\theta \quad (4)$$

Substituting $v \cos \theta$ for $r\Omega$ in the second term,

$$x_E - x_G = 2v\Omega\, dt \tag{5}$$

(normal to the direction of the velocity).

Dividing Eq. (5) by dt and designating $\dfrac{x_E - x_G}{dt}$ as a_C,

$$a_C = 2v\Omega \qquad \text{Formula XLVII}$$

The $2v\Omega$ of Formula XLVII is called the *Coriolis acceleration*. The component of acceleration directed toward the center of rotation of the frame of reference, $r\Omega^2$, is called the *centripetal acceleration* as before. The Coriolis acceleration is perpendicular to the velocity and to the left in a frame with counterclockwise rotation or to the right in a frame with clockwise rotation. These accelerations must be added vectorially to the acceleration of the moving point relative to the frame of reference to get its absolute acceleration. The proof of this proposition is tedious and unenlightening and is here omitted. The process of adding the accelerations may also be tedious, and it is usually simpler to take care of them by adding D'Alembert forces to the real forces of the system under study and so to reduce the problem to an equivalent system in a Newtonian frame of reference.

To take care of the centripetal acceleration, a fictitious force of $mr\Omega^2$ is assumed to act outward (from the center of rotation) on every particle of mass of the system under consideration and a force of $2mv\Omega$ laterally (to the right in a system rotating counterclockwise) on every particle of mass moving relative to the frame of reference. The first of these "forces" is properly called the *centrifugal force* when a rotating reference frame is used and the second is called the *Coriolis force*. In these expressions, m is the mass of the particle (in proper units), r its distance from the axis of rotation of the reference frame, v its speed relative to the frame, and Ω the angular velocity of rotation of the reference frame. Only the component of v in a plane perpendicular to the axis of rotation of the reference frame is used in computing the Coriolis force when the problem under consideration involves motions in three dimensions.

To acquire a better understanding of these two components of acceleration, the student should do the following exercises:

1. Repeat the derivation in the two special cases (*a*) where the velocity is radially outward in the reference frame and (*b*) where the

direction of the linear velocity is perpendicular to the line *OP* in Fig. 346,I.

2. Show that the y component of the vector CE is $-r\Omega^2\,dt \sin\theta$ and that the x component is $v\Omega\,dt + 2r\Omega^2\,dt\cos\theta + v\Omega\,dt\sin^2\theta$ and that these reduce to the centripetal and Coriolis accelerations.

Example 1

Figure 347 shows a smooth straight tube rotating in a horizontal plane about a vertical axis through O. A mass moving out through the tube has a velocity v_a at a distance a from O. Find the velocity v_b at the end of the tube at a distance b from O if the tube rotates uniformly at Ω rad per sec. Find also the pressure between the mass and the tube just before the mass emerges.

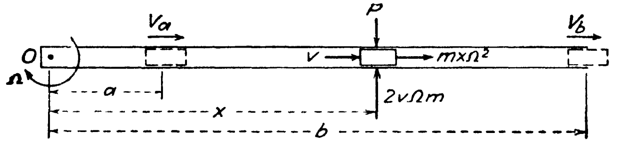

FIG. 347.

Let the reference frame revolve with the tube about the center O. The tube is then a straight radial line in the frame. A free-body diagram of the body at x distance from O shows a centrifugal force $mx\Omega^2$ acting outward along the tube, a Coriolis force $2mv\Omega$ acting laterally on the body, and a pressure P *exerted by the tube wall* laterally on the body. The weight of the body downward is balanced by an upward pressure from the tube wall. The body travels in a straight line and has no acceleration normal to the tube. Therefore, $P = 2mv\Omega$. The force $mx\Omega^2$ accelerates the mass along the tube. The acceleration is not constant,

$$mx\Omega^2 = m\frac{d^2x}{dt^2}$$

and it is easier to use an energy equation

$$\frac{mv_a^2}{2} + m\Omega^2\int_a^b x\,dx = \frac{mv_b^2}{2} \tag{1}$$

The integral is easily evalued as $\dfrac{m\Omega^2(b^2 - a^2)}{2}$, or the same result can be obtained by noting that the centrifugal force varies uniformly from a to b like the force of a linear spring so that the mean value of the initial and final centrifugal forces is the average value.

Evaluating the integral and solving for v_b,

$$v_b = \sqrt{v_a^2 + \Omega^2(b^2 - a^2)} \tag{2}$$

The force P equals the Coriolis force, and at b

$$P_b = 2m\Omega\sqrt{v_a^2 + \Omega^2(b^2 - a^2)} \tag{3}$$

Equations (2) and (3) are the answers to the problem. It should be noted that v_a and v_b are velocities *relative to the rotating frame of reference* and the "work" done by the centrifugal "force" is not the actual work done by the rotating tube on the small mass. The pressure P is a reality and the absolute difference of the kinetic energies of the mass at b and a represents work done by this force and is equal to $\int Px\Omega\, dt$, with P the same as the right-hand member of Eq. (3) with x substituted for b. The absolute velocity of the body at b is $\sqrt{v_b^2 + (b\Omega)^2}$.

The centrifugal force field is a potential field, as is gravity, and the work done by it is independent of the path of the body, so that the integral value in Eq. (1) is correct for any path if a and b are the radial distances from the center of the rotating frame to the beginning and end of the path. The student should prove this proposition to his own satisfaction.

Example 2

An airplane is flying at 600 mph along a straight horizontal path in the polar region. Find the angle at which the plane is banked against the Coriolis force.

The frame of reference is the earth, which rotates 2π rad in 24 hr (plus 2π rad per year neglected).

$$\Omega = \frac{2\pi}{24 \times 60 \times 60} = 0.0000727 \text{ rad per sec}$$

$$\text{Velocity} = \frac{600 \times 88}{60} = 880 \text{ ft per sec}$$

$$\text{Coriolis force} = m2v\Omega = \frac{2W\,880 \times 0.0000727}{g} = 0.0004W$$

where W is the weight of the plane. This is the horizontal component of the "lift." The vertical component is the weight. Angle of bank is $\tan^{-1} 0.0004 = 13.8'$. This small angle would result in an error of about 15 miles if it were neglected in calculating the position from astronomical observation.

Problems

1. In so-called "amusement parks" are to be found numerous devices for subjecting the human body to accelerations to which it is not accustomed. One of these devices is a large disk turning uniformly in a horizontal plane on which people may stand, walk, fall, and slide. In this problem and the several below, assume the wheel to be turning counterclockwise at 0.5 rad per sec and all other data to be relative to the wheel as a frame of reference. A 120-lb boy runs at 12 ft per sec along a diameter. Assume uniform speed and find the components of friction force on his foot when he is 12 ft from the center of the wheel.
Ans. $360/g$ lb toward the center and $1440/g$ lb toward the boy's left.
2. Find the coefficient of friction necessary for the motion of Prob. 1. *Ans.* 0.384.
3. After the boy of Prob. 1 has gone 4 ft beyond the center of the wheel, he starts to stop. How rapidly can he slow down if the friction coefficient under his foot is 0.5? *Ans.* 9.66 ft per sec per sec.

4. At what angle does he lean from the vertical as he passes the center of the wheel at 12 ft per sec? *Ans.* 20.5° to his left.
5. The boy of Prob. 1 walks at 6 ft per sec in a circle of radius 8 ft and center at the center of rotation of the wheel. What is the necessary friction force to keep him in his path when going (*a*) clockwise and (*b*) counterclockwise? *Ans.* (*a*) $60/g$ lb; (*b*) $1500/g$ lb.
6. Find the work which the boy on the wheel above must do to walk to the center of the wheel from a point 10 ft away. *Ans.* $1500/g$ ft-lb.
7. During all the action of the problems above, the wheel is rotated uniformly by some sort of driving mechanism. At times this mechanism is doing work on the wheel and at times it acts as a brake and absorbs work. Find the work done or absorbed by the mechanism during the action of Prob. 6. *Ans.* $3000/g$ ft-lb.
8. Note that the work of Prob. 7 was transmitted from the boy to the wheel by a tangential component of the friction force which the boy exerted on the wheel. This component was necessary to balance the Coriolis force. Assume any constant velocity for the walk of Prob. 6 and check the answer to Prob. 7 by multiplying the tangential component of foot friction by the absolute distance through which it acted.
9. The boy of Prob. 1 starts a piece of smooth ice sliding from the center of the wheel with an initial velocity of 12 ft per sec. Find the velocity of the ice relative to the wheel when it is 10 ft from the center, and find the radius of curvature of its path on the wheel at this time. *Ans.* 13 ft per sec; 3.61 ft.

   NOTE: Since the friction is zero, the wheel simply supports the ice and has no effect on its absolute velocity. Check the first answer by this simple consideration.
10. A boy standing on the floor outside the wheel of Prob. 1 places a small body on the wheel 12 ft from its center and holds it in place against the force of friction of the wheel sliding under it. From his standpoint the body is in equilibrium. Viewed from the wheel as a frame of reference, the body is moving at 6 ft per sec along a circle of 12-ft radius. Show that the "forces" account for the motion.
11. When an airplane flies uniformly in a straight line the vertical component of the air forces acting on it, called the *lift*, equals the weight of the plane. Find the difference in the lift of an airplane weighing 100,000 lb flying east at the equator at 600 mph and that of the same plane with the same speed in the opposite direction. *Ans.* About 800 lb.
12. A mouse of weight W runs up the inclined member of Fig. 308. Assume his velocity to be 10 ft per sec and his acceleration to be $0.25g$ up the bar when he is 1 ft from the lower end. Find the components of force which he exerts on the bar if the frame is rotating uniformly at 5 rad per sec.
    Ans. $2.85W$ toward the lower end of the bar; $0.7W$ perpendicular to the bar; $2.5W$ laterally.

Example 3

Find the dynamic effect of a two-bladed propeller rotating at constant velocity ω on an airplane which is turning about its vertical axis at the angular velocity Ω. In Fig. 348 the X axis represents the longitudinal axis of the airplane and the propeller rotates in a plane perpendicular to this axis. Let ωt be the angle which the propeller makes with the XY plane. Two symmetrically placed particles of mass dm are indicated in the propeller, and on each of these arrows are shown to indicate the centrifugal force and the Coriolis force acting on the particle. The centrifugal forces shown are for a frame of reference rotating with the propeller. Each is equal to $r\omega^2 dm$ and they are in opposite directions outward from the hub, so they cancel out and produce no effect except for stresses in the propeller itself. The force $2\Omega v_y\,dm$ shown in the upper blade parallel to the X axis directed toward the rear and its counterpart in the lower blade directed toward the front are the Coriolis forces for the particles considered from the XY plane as a frame of referencer otating (clockwise as seen from above) with the angular velocity Ω. The Coriolis forces arethe forces considered in the discussion below. (Omitted from the figure are centrifugal forces forward parallel to the X axis for the rotating XY plane as a frame of reference. These have the value $x\Omega^2\,dm$ where x is the distance from the vertical axis through the center of gravity of the airplane to the particle. They balance out with similar centrifugal forces of other parts of the machine and so produce no effects except for small stresses in the whole machine.)

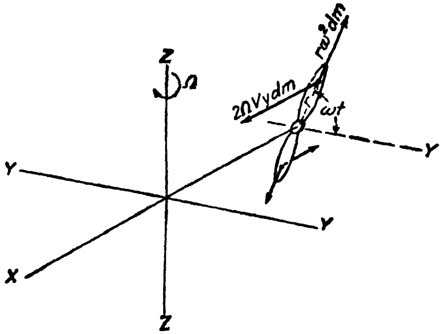

FIG. 348.

The two Coriolis forces shown in the propeller constitute a couple.

The Y component of the velocity of the particle shown in the upper blade is

$$v_y = r\omega \sin \omega t \tag{1}$$

The Coriolis force on this particle is

$$F_C = 2v_y\Omega\,dm = 2\Omega(r\omega \sin \omega t)\,dm \tag{2}$$

The moment with respect to the Y axis of all these forces in both blades is

$$M_Y = 2\omega\Omega \sin^2 \omega t \int r^2\,dm \tag{3}$$

The integral $\int r^2\,dm$ of Eq. (3) is the moment of inertia of the propeller with respect to an axis perpendicular to the X axis of Fig. 348 and to the line through the centers of gravity of the two blades of the propeller. For an ordinary two-bladed propeller it is very nearly the same as the moment of inertia of the propeller with respect to its axis of rotation. If $\sin^2 \omega t$ of Eq. (3) is replaced by $(1 - \cos 2\omega t)/2$, it will be seen that the torque about the Y axis varies sinusoidally from zero to a maximum value twice during each revolution of the propeller, reaching the maximum value each time the propeller is vertical. The solution of Prob. 13 below will show that the torque about the Z axis is a sinusoidal of half the magnitude and twice the frequency of that about the Y axis. These propeller effects sometimes set up serious resonant vibrations in component parts of the plane. If the student will visualize the Coriolis forces on a pair of elements as varying from zero at the horizontal position to a maximum at the vertical, the varying torques are easily understood.

Problems

13. Find the moment about the Z axis of the propeller of Example 3.

14. Show that if the propeller has three or four blades the torque during the turn is constant, and find its magnitude. *Ans.* $M_y = I\Omega\omega$; $M_z = 0$.

189. The Gyroscope. The three- and four-bladed propellers of Prob. 14 of the preceding article have a property in common with a solid homogeneous disk in that the moment of inertia of the propeller with respect to any axis in its plane and through its center of gravity is the same as that for any other similar axis. Such a propeller behaves dynamically like a spinning disk or wheel, and the problem could easily have been solved by a very useful formula called the *principal equation of the gyroscope.* Using a rotating frame of reference, this formula can be derived by taking moments about an axis in the rotating plane of reference and perpendicular to the axis of spin.

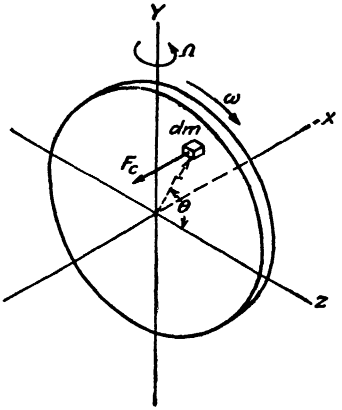

FIG. 349.

Figure 349 represents a disk element of a body spinning with a velocity ω about the X axis while the XZ plane rotates with an angular velocity Ω about the Y axis. The arrow shown on the small element dm of the disk represents the Coriolis force F_C for this element.

$$F_C = 2v_z\Omega\, dm = 2(r\omega \sin\theta)\Omega\, dm \qquad (1)$$

The moment about the Z axis of all such forces on the disk is $\int F_C r \sin\theta$.

$$M_z = 2\omega\Omega \int r^2 \sin^2\theta\, dm = I_x\omega\Omega \qquad (2)$$

Equation (2) is the principal equation of the gyroscope. It is usually written with M_z expressed as T (for torque) and with Ω as the dependent variable,

$$\Omega = \frac{T}{I\omega} \qquad \text{Formula XLVIII}$$

As is indicated by Formula XLVIII, if it is desired to turn a spinning disk about an axis in its plane, it is necessary to apply a torque about another axis in the plane at right angles to the axis of the desired turn.

A spinning disk or flywheel mounted on gimbals to turn freely in any direction is called a *gyroscope*. The word was coined to describe the ability of such a device to make visible the rotation of the earth. The gyroscope is a sensitive detector of angular motions and has found many useful applications in such devices as bank and turn indicators, artificial horizons, gyrocompasses, ship stabilizers, etc. Most of the useful applications can be understood in the light of Formula XLVIII although some of them, like the gyrocompass, are quite complicated. An attempt to turn a spinning wheel about an axis in its plane by a direct application of torque about that axis results in only a minute angular motion about that axis if the wheel is free to turn about the perpendicular axis. This property is called the *rigidity of the gyroscope.*

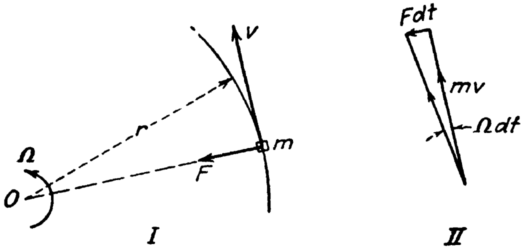

FIG. 350.

The steady angular velocity about one axis resulting from a steady torque about the other (both axes perpendicular to the spin axis) is called *precession.* A spinning wheel may be held up with the axis horizontal by a force not in line with the weight of the wheel if the wheel is allowed to precess.

The behavior of a gyroscope seems strange solely because of the unfamiliarity of the observer with the device. The behavior in all cases is analogous to and mathematically identical with the change of direction of linear momentum resulting from a force at right angles to the direction of the velocity. Figure 350,I shows a mass m moving uniformly along a curve and held in its path by the force F at right angles to the direction of the velocity, all in the well-known relation, $F = mr\Omega^2$. The momentum of this mass is in the direction of its velocity and equals, in magnitude, $mr\Omega$. Calling the momentum mv, the force relation above becomes $F = mv\Omega$. Rearranging, $\Omega = F/mv$, which is exactly like Formula XLVIII, with torque replaced by force and moment of momentum replaced by momentum. Figure 350,II is a vector diagram from which the relation of change of direction of momentum to the force and momentum involved can be easily derived. From the geometry of the figure, $mv\Omega\,dt = F\,dt$, if dt is infinitesimal.

When it is desired to deflect a moving body from its path, it is necessary to apply a force in the direction of the desired deflection. The directional relation is not so obvious when it is desired to turn the gyroscope about an axis perpendicular to the axis of spin. The proper sense for the applied torque can be remembered by the *carry forward rule.* The torque is imagined to be applied by two forces acting on the spinning disk, and the disk is imagined to drag the points of application *forward* one quarter turn. Then the direction of the torque will indicate the direction of the precession. The student should consider this rule in connection with the derivation of Formula XLVIII until he sees how it applies.

Problems

1. Using a diagram like Fig. 350,II, with force replaced by torque and momentum replaced by angular momentum, derive Formula XLVIII. (Use the right-hand screw convention for the directions of the vectors with the angular momentum vector parallel to the X axis in a figure like Fig. 349.)
2. A disk 6 in. in diameter is mounted on a light axle which is supported in a horizontal position by two ball bearings, each 3 in. from the center of the disk. The bearings are held by two vertical strings while the disk revolves at 3000 rpm. If one of the strings is cut, what will be the rate of precession? *Ans.* 7.82 rpm.
3. If the axis of the disk of Prob. 2 was pointed northward and the spin was clockwise as viewed from the south when the north string was cut, what was the direction of precession? *Ans.* Counterclockwise as viewed from above.
4. A turbine rotor weighing 3216 lb with a radius of gyration of 2 ft is mounted in bearings, fore and aft, 4 ft apart. If the ship pitches with an angular velocity of 0.5 rad per sec while the rotor turns at 3600 rpm, what lateral force will be induced at the bearings? *Ans.* 18,850 lb.

APPENDIX

SOME CATENARY PROBLEMS

190. Difficult Forms. Formula X of Chap. XI gives a direct solution for x when y, s, and c are known. Formula IX gives a relation between y, s, and c which makes it convenient to calculate any one of these if the other two are given. Formulas XI and XIII give y in terms of x and c, and Formulas XII and XIV give s in terms of x and c. Since the sag is $y - c$, if the sag is known and one of the three quantities y, s, or c is given, the other two may be calculated for use in these formulas.

There are a number of cases in which c is not given and cannot be conveniently calculated. Some of these cases are length and span given, span and resultant tension given, span and sag given. Since c occurs twice in each of Formulas X, XI, XII, XIII, and XIV, it is the most difficult term to calculate from these formulas.

191. Length and Span Given. Formula XIV gives

$$s = c \sinh \frac{x}{c} \qquad \text{Formula XIV}$$

If both sides of this formula are divided by x.

$$\frac{s}{x} = \frac{c}{x} \sinh \frac{x}{c} \tag{1}$$

$$\frac{s}{x} = \frac{\sinh (x/c)}{x/c} \tag{2}$$

A problem of this kind may be solved conveniently by means of a table which gives $(\sinh x)/x$. Table 2 gives these functions from $x = 0.080$ to $x = 0.120$.*

Example

A chain 200 ft long is stretched between points at the same level 199.572 ft apart. Find the horizontal tension in terms of the weight per foot and find the sag.

$$\frac{s}{x} = \frac{100}{99.786} = 1.0021446 = \frac{\sinh (x/c)}{x/c}$$

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* Complete tables of $(\sinh x)/x$, $(\cosh x)/x$, and $(\cosh x - 1)/x$ are to be found in Campbell's "Numerical Tables," Houghton Mifflin Company.

From Table 2,

x	$(\sinh x)/x$	$(\sinh x)/x$
0.113	1.0021295	1.0021295
0.114	1.0021674	
$\frac{x}{c}$		1.0021446
	379	151

$$\frac{x}{c} = 0.113 + 0.001 \times \frac{151}{379} = 0.113398$$

$$c = \frac{99.786}{0.113398} = 879.96 \text{ ft}$$

$$y = c \cosh 0.113398 = 879.96 \times 1.0064366 = 885.62 \text{ ft}$$
$$y - c = 5.66 \text{ ft}$$

For a check, $y^2 = c^2 + 10{,}000 = 784{,}329.60$

$$y = 885.62 \text{ ft}$$

192. Span and Total Tension Given. When the resultant tension is given, the value of y is found from $T = wy$.

$$y = c \cosh \frac{x}{c} \tag{1}$$

$$\frac{y}{x} = \frac{\cosh (x/c)}{x/c} \tag{2}$$

Example 1

A cable is stretched between points at the same level which are 200 ft apart. The total tension is equal to the weight of 1000 ft of cable. Find the horizontal tension and the length of the cable.

$$\frac{y}{x} = 10$$

From Table 2,

x	$(\cosh x)/x$	$(\cosh x)/x$
0.100	10.050042	10.050042
0.101	9.951533	
$\frac{x}{c}$		10.
	0.098509	0.050042

$$\frac{x}{c} = 0.100 + 0.001 \frac{50{,}042}{98{,}509} = 0.100508$$

$$c = \frac{100}{0.100508} = 994.95$$

$$\text{Sag} = y - c = 5.05 \text{ ft}$$
$$s = c \sinh 0.100508 = 994.95 \times 0.1006773 = 100.169 \text{ ft}$$
$$2s = 200.338 \text{ ft}$$

For a check, $s^2 = 1{,}000{,}000 - 989{,}925.50 = 10{,}074.50$

$$s = 100.37 \text{ ft}$$

TABLE 2. HYPERBOLIC FUNCTIONS

x	$\sinh x$	$\frac{\sinh x}{x}$	$\cosh x$	$\frac{\cosh x - 1}{x}$	$\frac{\cosh x}{x}$	$\tanh x$
0 080	0 0800854	1 0010670	1 0032017	0 04002134	12 540021	0 0798298
081	10886	0939	2823	052214	386201	0808233
082	20919	1210	3639	102298	236145	18167
083	30953	1486	4465	152383	089717	28099
.084	40988	1764	5301	202470	11 946787	38030
085	0851024	1 0012046	1 0036147	04252559	11 807231	0847959
086	61060	2331	7003	302651	670933	57886
087	71098	2620	7869	352744	537780	67812
088	81136	2912	8745	402840	407665	77735
089	91175	3207	9631	452938	280484	87658
090	0901215	1 0013505	1 0040527	04503038	11 156141	0897578
091	11256	3807	1434	553141	034542	0907496
092	21298	4113	2350	603245	10 915598	17413
093	31341	4421	3276	653352	799222	27328
094	41385	4733	4213	703462	685332	37241
095	0951430	1 0015048	1 0045159	04753573	10 573852	0947152
096	61475	5367	6115	803688	464704	57061
097	71522	5689	7082	853804	357816	66969
098	81569	6014	8058	903923	253121	76875
099	91618	6343	9045	954044	150551	86778
100	1001668	1 0016675	1 0050042	05004168	10 050042	0996680
101	11718	7010	1048	054294	9 951533	1006580
102	21770	7349	2065	104423	854966	16477
103	31822	7691	3092	154555	760283	26373
104	41876	8036	4129	204689	667432	36267
105	1051930	1 0018385	1 0055176	05254825	9 576358	1046158
106	61986	8737	6233	304964	487012	56048
107	72043	9093	7300	355106	399345	65935
108	82101	9451	8377	405251	313312	75820
109	92160	9813	9464	455398	228866	85704
110	1102220	1 0020179	1 0060561	05505548	9 145965	1095585
111	12281	0548	1668	555701	064566	1105464
112	22343	0920	2786	605856	8 984630	15340
113	32406	1295	3913	656015	906118	25215
114	42471	1674	5050	706176	828992	35087
115	1152536	1 0022056	1 0066198	05756340	8 753216	1144957
116	62603	2442	7355	806507	678755	54825
117	72671	2831	8523	856676	605575	64690
118	82740	3223	9701	906849	533645	74554
119	92811	3618	70889	957025	462932	84414
120	1202882	1 0024017	1 0072086	.06007203	8 393405	1194273

The values for s do not check. The result from $c \sinh (x/c)$ is the better. In the calculation of the small side of a right triangle from the difference of the squares of the hypotenuse and a side nearly as large, a small error in one of these makes a large error in the length of the short side. The error in c was made in interpolating from the table. The curve which represents $(\cosh x)/x$ differs so much from a straight line that a linear interpolation to three significant figures is not warranted. Interpolation of $(\cosh x)/x$ between $x = 0.100$ and 0.101 by Stirling's method gives

x	Stirling's method	Linear interpolation
0.1001	10.040102	10.040191
0.1002	30182	30340
0.1003	20282	20489
0.1004	10401	10638
0.1005	00542	00788
0.1006	9.990701	9.990937
0.1007	80880	81086
0.1008	71078	71235
0.1009	61296	61384

Interpolation between 0.1005 and 0.1006 as computed by Stirling's method gives

$$\frac{x}{c} = 0.1005 + 0.0001 \times \frac{542}{9841} = 0.1005055$$

$$c = \frac{100}{0.1005055} = 994.970$$

which differs very little from the previous result.

$$s = 994.97 \sinh 0.1005055 = 994.97 \times 0.1006748 = 100.1684 \text{ ft}$$

This value of s differs very little from the previous result obtained from the sinh table.

$$s^2 = 1{,}000{,}000 - 889{,}965.30 = 10{,}034.70$$

$$s = 100.173 \text{ ft}$$

If 944.971 is taken as the value of c,

$$y^2 - c^2 = 10{,}032.66 \text{ and } s = 100.163 \text{ ft}$$

Example 2

A uniform cable is stretched between points at the same level. What is the ratio of the length of cable to the span when the resultant tension at the ends is the minimum?

From Campbells "Numerical Tables," cosh/x has the minimum value of 1.5089 when $x = 1.20$

$$\frac{x}{c} = 1.20 \qquad c = 0.8333x$$

$$\frac{y}{x} = 1.5089 \qquad y = 1.5089x$$

If θ is the slope of the cable at the ends, $\cos\theta = 8333/15{,}089$

$$\begin{aligned} \cos\theta &= 0.5523 \\ \theta &= 56°28' \\ s &= y \sin 56°28' \\ s &= 0.83356 \times 1.5089x \doteq 1.2578x \\ \text{Sag} &= 1.5089x - 0.8333x = 0.6756x = 0.3378 \text{ span} \end{aligned}$$

193. Span and Sag Given. The sag is $y - c$, which is

$$\text{Sag} = c\left(\cosh\frac{x}{c} - 1\right) \tag{1}$$

$$\frac{\text{sag}}{x} = \frac{\cosh(x/c) - 1}{x/c} \tag{2}$$

Example

A rope stretched between points at the same level 200 ft apart sags 5 ft 3 in. at the middle. Find the length of the rope.

$$\frac{\cosh(x/c) - 1}{x/c} = 0.0525$$

From the table, by interpolation,

$$\begin{aligned} \frac{x}{c} &= 0.104904 \\ c &= 953.25 \\ \sinh 0.104904 &= 0.1050965 \\ s &= 953.25 \times 0.1050965 = 100.183 \\ 2s &= 200.366 \text{ ft} \end{aligned}$$

194. Solution by Infinite Series. The series for sinh x is

$$\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} + \cdots \tag{1}$$

Equation (1) is the same as the series for the sine of x, except that all the terms are positive. In the series for the sine of x the terms are alternately positive and negative.

When x/c is substituted for x in Eq. (1) and the series is multiplied by c, the result is

$$s = c\left(\frac{x}{c} + \frac{x^3}{\underline{|3}\,c^3} + \frac{x^5}{\underline{|5}\,c^5} + \frac{x^7}{\underline{|7}\,c^7} + \cdots\right) \tag{2}$$

$$s = x\left(1 + \frac{x^2}{\underline{|3}\,c^2} + \frac{x^4}{\underline{|5}\,c^4} + \frac{x^6}{\underline{|7}\,c^6} + \cdots\right) \tag{3}$$

The series for cosh x is

$$\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} + \cdots \tag{4}$$

This is the same as the series for the cosine of x, except that all the terms are positive for cosh x, while they are alternately positive and negative for cos x.

When x/c is substituted for x in Eq. (4) and the series is multiplied by c, the result is

$$y = c\left(1 + \frac{x^2}{\underline{|2}\,c^2} + \frac{x^4}{\underline{|4}\,c^4} + \frac{x^6}{\underline{|6}\,c^6} + \cdots\right) \tag{5}$$

$$y - c = \frac{x^2}{\underline{|2}\,c^2} + \frac{x^4}{\underline{|4}\,c^4} + \frac{x^6}{\underline{|6}\,c^6} + \cdots \tag{6}$$

When the load is uniform for each unit of horizontal distance, as in Art. 104, the equation of the curve is

$$y = \frac{x^2}{2c'} \tag{7}$$

in which y is the deflection upward from the tangent at the lowest point and c' is the length whose weight is equal to the horizontal tension. This is the equation of a parabola with the axis vertical.

To find the length of the parabola,

$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \left[1 + \left(\frac{x}{c'}\right)^2\right]^{\frac{1}{2}} dx \tag{8}$$

Expanding Eq. (8) by the binomial theorem,

$$ds = \left(1 + \frac{x^2}{2c'^2} - \frac{x^4}{8c'^4} + \frac{x^6}{16c'^6} - \frac{5x^8}{128c'^8}\right) dx \tag{9}$$

Integrating Eq. (9),

$$s = x + \frac{x^3}{6c'^2} - \frac{x^5}{40c'^4} + \frac{x^7}{112c'^6} - \frac{5x^9}{1152c'^8} + \cdots \tag{10}$$

A comparison of Eq. (10) for the parabola with Eq. (3) for the catenary shows that the first two terms are identical if $c = c'$. The coefficient of the third term of Eq. (3) is $\frac{1}{120}$ and that of the third term of Eq. (10) is $-\frac{1}{40}$. The coefficients of the third term differ by $\frac{1}{30}$. When x/c is relatively small, $\frac{1}{30}(x/c)^4$ becomes so small that its value may be neglected from all but the most accurate calculations. Also the first term of Eq. (6) for the deflection of the catenary is identical with Eq. (7). There are very few problems to which the parabola equation strictly applies. It is frequently used as an approximation for the catenary. When so used, it is better to consider the parabola as the first term of Eq. (6), rather than to assume that the load is uniformly distributed for each unit of horizontal distance.

195. The Gudermannian. The gudermannian of x is defined by the relations

$$\begin{aligned} \sin \operatorname{gd} x &= \tanh x \\ \tan \operatorname{gd} x &= \sinh x \\ \sec \operatorname{gd} x &= \cosh x \end{aligned}$$

For the catenary the angle θ (Fig. 226) which the tangent to the curve at any point makes with the horizontal is the gudermannian of x/c. From Fig. 226,*V*,

$$\begin{aligned} \sin \theta &= \frac{s}{y} = \frac{c \sinh (x/c)}{c \cosh (x/c)} = \tanh \frac{x}{c} \\ \tan \theta &= \frac{s}{c} = \frac{c \sinh (x/c)}{c} = \sinh \frac{x}{c} \\ \sec \theta &= \frac{y}{c} = \frac{c \cosh (x/c)}{c} = \cosh \frac{x}{c} \end{aligned}$$

An excellent gudermannian table is found in the Smithsonian Mathematical Tables—Hyperbolic Functions. The angles are expressed in radians to the seventh decimal place and in degrees, minutes, and seconds to one-hundredth of a second. From 0 to 3 the values of x differ by 0.001, and from 3 to 6 by 0.01. The differences of the second order are so small and uniform that linear interpolations are accurate to tenths of a second.

By means of a gudermannian table and a seven-place trigonometric table, calculations of the catenary may be made to any desired degree of accuracy.

Example

Solve Example 1 of Art. 107. In this example $c = 500$, $x = 60$, and $x/c = 0.12$.

$$\text{Gudermannian of } 0.12 = 6°51'32''.59$$

$$y = \frac{500}{\cos 6°51'32''.59} = \frac{500}{0.99284} = 503.605 \text{ ft}$$

$$\text{Sag} = 3.605 \text{ ft}$$

$$s = 500 \tan 6°51'32''.59 = 500 \times 0.12029 = 60.145 \text{ ft}$$

196. Reversed Series. Equations (3) and (6) of Art. 194 give length and sag in terms of x and c. These may be found easily by means of the tables of hyperbolic functions. The troublesome problems, as shown in Arts. 190 to 193, are those in which c is not given and cannot be calculated by Formula IX. When tables of $(\sinh x)/x$, $(\cosh x)/x$, and $(\cosh x - 1)/x$ are not available or are not sufficiently exact, many problems may be solved by means of infinite series which give x/c in terms of other dimensions of the catenary.

When the sag and the span are given,

$$\frac{y-c}{x} = \frac{\cosh(x/c) - 1}{x/c} = \frac{\cosh z - 1}{z} = \frac{z}{\underline{|2}} + \frac{z^3}{\underline{|4}} + \frac{z^5}{\underline{|6}} + \frac{z^7}{\underline{|8}} + \cdots \qquad (1)$$

in which $z = x/c$. If $(y - c)/x$ is represented by v,

$$v = \frac{z}{\underline{|2}} + \frac{z^3}{\underline{|4}} + \frac{z^5}{\underline{|6}} + \frac{z^7}{\underline{|8}} + \frac{z^9}{\underline{|10}} + \frac{z^{11}}{\underline{|12}} + \frac{z^{13}}{\underline{|14}} + \cdots \qquad (2)$$

When the series of Eq. (2) is reversed by the method of undetermined coefficients (see any advanced algebra), the result is

$$z = 2v - \frac{2v^3}{3} + \frac{26v^5}{45} - \frac{622v^7}{945} + \frac{5016v^9}{6300} - \cdots \qquad (3)$$

Example

Find x/c for the example of Art. 193 by the series. Calculate the length and check. In this example $v = 0.0525$,

$$x/c = z = 0.1050 - 0.00009647 + 0.00000023 = 0.10490376$$

$$\log\frac{x}{c} = \bar{1}.0207910$$

$$\log x = \underline{2.}$$

$$\log c = 2.9792090$$

$$c = 953.2549$$

Using the gudermannian from Smithsonian Mathematical Tables of Hyperbolic Functions to secure the maximum accuracy,

$$\text{gd } 0.10490376 = \theta = 5°59'58''.39$$

$$\log \sec\theta = 0.0023853$$

$$\log c = \underline{2.9792090}$$

$$\log y = 2.9815943$$

$$y = 958.5049 \text{ ft}$$

$$c = \underline{953.2549 \text{ ft}}$$

$$y - c = 5.2500 \text{ ft}$$

The results check to the seventh significant figure for y and c and to the fifth significant figure for the sag.

$$\begin{aligned}
\log\tan\theta &= 9.0215876\\
\log c &= \underline{2.9792090}\\
\log s &= 2.0007966\\
s &= 100.18359 \text{ ft}\\
y^2 - c^2 &= 10036.73895 = s^2\\
s &= 100.18353 \text{ ft}
\end{aligned}$$

When the length and the span are given,

$$\frac{s}{x} = \frac{\sinh (x/c)}{x/c} = \frac{\sinh z}{z} = 1 + \frac{z^2}{\underline{|3}} + \frac{z^4}{\underline{|5}} + \frac{z^6}{\underline{|7}} + \cdots \tag{4}$$

If $(s/x) - 1$ is represented by w,

$$w = \frac{z^2}{\underline{3}} + \frac{z^4}{\underline{|5}} + \frac{z^6}{\underline{|7}} + \frac{z^8}{\underline{|9}} + \frac{z^{10}}{\underline{|11}} + \frac{z^{12}}{\underline{|13}} + \cdots \tag{5}$$

When the series of Eq. (5) is reversed, the result is

$$z^2 = 6w - 1.8w^2 + \frac{576w^3}{700} - \frac{312w^4}{700} + \cdots \tag{6}$$

Example

Solve the example of Art. 191 for the sag and total tension and check.

$$\frac{s}{x} - 1 = \frac{100}{99.786} - 1 = 0.00214459 = w$$

$$z^2 = 0.01286754 - 0.00000828 + 0.00000001 = 0.01285927$$

$$\begin{aligned}
\log z^2 &= \bar{2}.1092162\\
\log z &= \bar{1}.0546081\\
\frac{x}{c} = z &= 0.1133987\\
\log c &= 2.9444615\\
c &= 879.957 \text{ ft}
\end{aligned}$$

$$\begin{aligned}
\text{gd. } 0.1133987 = \theta &= 6°29'00''.19\\
\log\sec\theta &= 0.0027863\\
\log c &= \underline{2.9444615}\\
\log y &= 2.9472478\\
y &= 885.621 \text{ ft}\\
y - c &= 5.664 \text{ ft}
\end{aligned}$$

To find s for a check,

$$\begin{aligned}
\log\tan\theta &= 9.0555385\\
\log c &= \underline{2.9444615}\\
\log s &= 2.0000000\\
s &= 100.0000
\end{aligned}$$

When the span and the total tension are given, y is calculated by dividing the tension by the weight per unit length.

$$\frac{y}{x} = \frac{\cosh (x/c)}{x/c} = \frac{\cosh z}{z} \tag{7}$$

Since y is greater than s and s is greater than x, y is always greater than x. The reversed series for finding z should be in terms of x/y to converge rapidly.

$$\frac{x}{y} = \frac{z}{\cosh z} \tag{8}$$

The division of z by the series for cosh z gives

$$\frac{x}{y} = u = z - \frac{z^3}{\underline{|2}} + \frac{5z^5}{\underline{|4}} - \frac{61z^7}{\underline{|6}} + \frac{1385z^9}{\underline{|8}} - \frac{50521z^{11}}{\underline{|10}} + \cdots \tag{9}$$

Reversion of Eq. (9) gives

$$z = u + \frac{u^3}{2} + \frac{13u^5}{24} + \frac{541u^7}{720} + \frac{70345u^9}{40320} + \cdots \tag{10}$$

Example 3

Solve Example 1 of Art. 192 for the length and sag.

$$\begin{aligned}
\frac{x}{y} &= 0.1 = u \\
u &= 0.1 \\
&\quad + 0.0005 \\
&\quad + 0.000005417 \\
&\quad \underline{+ 0.000000070} \\
\frac{x}{c} &= 0.100505487
\end{aligned}$$

which checks Stirling's interpolation of Table 2 as given in Art. 192.

$$\begin{aligned}
\log 0.10050549 &= 1.0021898 \\
\log\ \ 100 &= \underline{2} \\
\log c &= 2.9978102 \\
c &= 994.9705 \\
\text{gd}\ 0.10050549 &= 5°44'55''.94 \\
\log \sec \theta &= 0.0021898 \\
\log c &= \underline{2.9978102} \\
\log y &= 3.0000000 \\
y &= 1000. \\
\log \tan \theta &= 9.0029210 \\
\log c &= \underline{2.9978102} \\
\log s &= 2.0007312 \\
s &= 100.1685
\end{aligned}$$

For a check,

$$s^2 = 1{,}000{,}000 - 989{,}966.29587 = 10033.70413$$
$$s = 100.1684$$

Problems

Calculate x/c by the suitable series, then use any available table of hyperbolic functions. The answers given were found by means of the gudermannian and seven-place tables of logarithmic trigonometric functions.

1. A cable weighing 2.4 lb per ft is stretched between points at the same level 200 ft apart and sags 10 ft. Find the tension at the ends and the length of the cable.
   Ans. $x/c = 0.199339$; $c = 501.658$ ft; $\theta = 11°20'47'.0$; $y = 511.658$ ft; $s = 100.6636$ ft.
2. Solve Prob. 1 for a sag of 20 ft.
   Ans. $x/c = 0.3948435$; $\theta = 22°03'25''.1$; $c = 253.2649$ ft; $y = 273.2649$ ft; $s = 102.6187$ ft.
3. A cable is stretched between points at the same level 200 ft apart. The resultant tension at each end is equal to the weight of 500 ft of cable. Find the length and the sag.
   Ans. $x/c = 0.2041835$; $\theta = 11°37'06''.25$; $c = 489.7555$ ft; $s = 100.696$ ft.
4. A cable 521 ft long is stretched between points at the same level 500 ft apart. Find the sag. *Ans.* 63.66 ft.
5. A cable 540 ft long is stretched between points at the same level 500 ft apart. Find the sag. Since the series converges too slowly, use Campbell's table for $(\sinh x)/x$.

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